Trajectory Optimization for a Robotic Arm

Team 8

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Abstract

Robotic arms are widely used in pick-and-place operations in many industries and most companies have a financial interest in optimizing the energy they expend and the speed they operate. The work presented herein describes a nonlinear programming approach to trajectory optimization of a 3-link robotic arm of KUKA youBot in 2D workspace. These types of robots are widely used in various industries where pick-and-place operations are necessary, such as mechanical assembly facilities and shipping warehouses.

The optimization algorithm seeks to minimize a multiobjective cost function balancing both energy expended and time taken. While the problem may not be convex, we feel confident we have found global minima by utilizing a randomized multistart approach. We have also studied variations of the objective function weighting and presented the Pareto frontier; companies could tailor this weight to their needs and select an optimal trajectory that best meets their needs.The trade-off between the time and the work expended functions is significant at the extreme ends of the weighted sum. Still, in general, most of the trade-offs between the two objective functions are insignificant for the majority of cases.

Introduction

The goal of this project is to explore trajectory optimization of a 3-link robotic arm in 2 dimensions. We have taken inspiration from the KUKA youBot - a robotic arm with a mobile base. This project looks into deciding the optimal trajectory for the robotic arm based on the multi-objective of minimizing work done and the time spent in reaching the destination using nonlinear programming and trapezoidal collocation.

The answer to this problem is not something that can easily be determined by manual calculation. A trajectory consists of many discrete time steps, each of which has position, velocity, and input force information; these become the decision variables of the nonlinear program. There is no simple way to analytically formulate such an optimal trajectory, especially not one that strives to minimize the weighted objectives of time and energy.

Tradeoffs must be made in the optimal solution between these two objectives. Additionally, the trajectory is constrained to follow practical bounds such as joint limits and torque limits while also satisfying laws of physics. The dynamic equations of motion for a multi-link robotic system can be complex.

Problem Statement

The objective of the problem statement can be formally defined as a weighted minimization of work expended and time taken in the completion of a predefined task. The hypothetical continuous trajectory is discretized using trapezoidal collocation to limit the decision variables to a discrete set of numbers. [1] The resultant trajectory is subject to practical bounds and must follow the dynamic equations of motion for the robot. The problem can be formulated in the standard negative null form as follows:

$\boldsymbol{minimize}\;\; f(\boldsymbol{x}) = \beta n\Delta t + (1-\beta)\sum_{k=1}^{n-1}\Delta t\,\frac{w_{k+1}+w_k}{2}$	weighted sum of time taken, and work expended
$\boldsymbol{w.r.t}\;\; \boldsymbol{x} = \left[\cdots \;\; (\mathbf{x}_k^{\mathrm{P}})^{\top} \;\; (\mathbf{x}_k^{\mathrm{V}})^{\top} \;\; (\mathbf{x}_k^{\mathrm{F}})^{\top} \;\; \cdots \;\; \Delta t\right]^{\top} \forall k \in \{1, \ldots, n\}$	positions, velocities, input forces, and time step size
$\boldsymbol{s.t.}\;\; \boldsymbol{x}_{k+1}^{\mathrm{V}} = \boldsymbol{x}_k^{\mathrm{V}} + \Delta t\left(\frac{\boldsymbol{\alpha}_{k+1}+\boldsymbol{\alpha}_k}{2}\right) \forall k \in \{1, \ldots, n-1\}$	trapezoidal collocation of velocity
$\boldsymbol{x}_{k+1}^{\mathrm{P}} = \boldsymbol{x}_k^{\mathrm{P}} + \Delta t\left(\frac{\boldsymbol{x}_{k+1}^{\mathrm{V}}+\boldsymbol{x}_k^{\mathrm{V}}}{2}\right) \quad \forall k \in \{1, \ldots, n-1\}$	trapezoidal collocation of position
$\boldsymbol{x}_0^{\mathrm{P}} - \boldsymbol{x}_{\mathrm{I}}^{\mathrm{P}} = \mathbf{0}\,;\, \boldsymbol{x}_n^{\mathrm{P}} - \boldsymbol{x}_{\mathrm{F}}^{\mathrm{P}} = \mathbf{0}$	initial and final position constraint
$\boldsymbol{x}_0^{\mathrm{V}} - \boldsymbol{x}_{\mathrm{I}}^{\mathrm{V}} = \mathbf{0}\,;\, \boldsymbol{x}_n^{\mathrm{V}} - \boldsymbol{x}_{\mathrm{F}}^{\mathrm{V}} = \mathbf{0}$	initial and final velocity constraint
$\boldsymbol{x}_k^{\mathrm{F}} - \boldsymbol{x}_{\mathrm{U}}^{\mathrm{F}} \leq \mathbf{0}\,;\, \boldsymbol{x}_{\mathrm{L}}^{\mathrm{F}} - \boldsymbol{x}_k^{\mathrm{F}} \leq \mathbf{0}$	upper and lower force bounds
$\boldsymbol{x}_k^{\mathrm{P}} - \boldsymbol{x}_{\mathrm{U}}^{\mathrm{P}} \leq \mathbf{0}\,;\, \boldsymbol{x}_{\mathrm{L}}^{\mathrm{P}} - \boldsymbol{x}_k^{\mathrm{P}} \leq \mathbf{0}$	upper and lower position bounds
$\boldsymbol{x}_k^{\mathrm{V}} - \boldsymbol{x}_{\mathrm{U}}^{\mathrm{V}} \leq \mathbf{0}\,;\, \boldsymbol{x}_{\mathrm{L}}^{\mathrm{V}} - \boldsymbol{x}_k^{\mathrm{V}} \leq \mathbf{0}$	upper and lower velocity bounds
$\boldsymbol{where}\;\; \boldsymbol{\alpha}_k = \boldsymbol{M}_k^{-1}(\mathbf{x}_k^{F} - \boldsymbol{C}_k\boldsymbol{x}_k^{V} - \boldsymbol{N}_k)$	dynamic equations of motion for the robot
$\mathbf{M}_k\left(\mathbf{x}_k^{\mathrm{P}}, \mathbf{m}_{\mathrm{R}}^{\mathrm{L}}, \mathbf{m}_{\mathrm{R}}^{\mathrm{XX}}, \mathbf{d}_{\mathrm{R}}, \boldsymbol{\theta}_{\mathrm{R}}\right)$	mass matrix of the robot
$\mathbf{C}_k\left(\mathbf{x}_k^{\mathrm{P}}, \mathbf{x}_k^{\mathrm{V}}, \mathbf{m}_{\mathrm{R}}^{\mathrm{L}}, \mathbf{m}_{\mathrm{R}}^{\mathrm{XX}}, \mathbf{d}_{\mathrm{R}}, \boldsymbol{\theta}_{\mathrm{R}}\right)$	Coriolis matrix of the robot
$\mathbf{N}_k\left(\mathbf{x}_k^{\mathrm{P}}, \mathbf{m}_{\mathrm{R}}^{\mathrm{L}}, \mathbf{d}_{\mathrm{R}}, \boldsymbol{\theta}_{\mathrm{R}}\right)$	non-linear terms matrix of the robot
$w_k = \left(\boldsymbol{x}_k^{\mathrm{F}}\right)^{T}\boldsymbol{x}_k^{\mathrm{F}}$	work function

The full list of decision variables and parameters along with their values can be found in Appendix A. Definitions for the mass matrix, Coriolis matrix, and nonlinear-term matrices can be found in Appendix B. [2]

Analysis of Problem Statement

Classification and Scale of the Problem

The problem is a multiobjective NLP. While the problem itself is defined within the continuous-time domain, it is discretized through trapezoidal collocation so that every decision variable is rendered discrete. Each additional collocation point that is considered adds 9 decision variables to the problem. The timestep itself is included along with the position, velocity, and torques of each link of the robotic arm to form the complete set of variables. The total number of decision variables is therefore $9n + 1$, where n is the total number of collocation points.

The negative-null form above shows that we have $6(n-1)+12$ total equality constraints - 12 for both the starting and ending positions, and 6 for each collocation point. The latter also encompass all the dynamics of the problem. In addition, we have $18n+2$ inequality constraints that act as simple bounds for the position, velocity, and torques of each arm link throughout the trajectory. These bounds are derived directly from youBot parameters as detailed in documentation about the robot.

For reference, the image below illustrates the available workspace for the youBot Arm, with the shaded gray area indicating points which the end-effector can successfully reach. The orange outline represents the outer limits of the end-effector's movement.

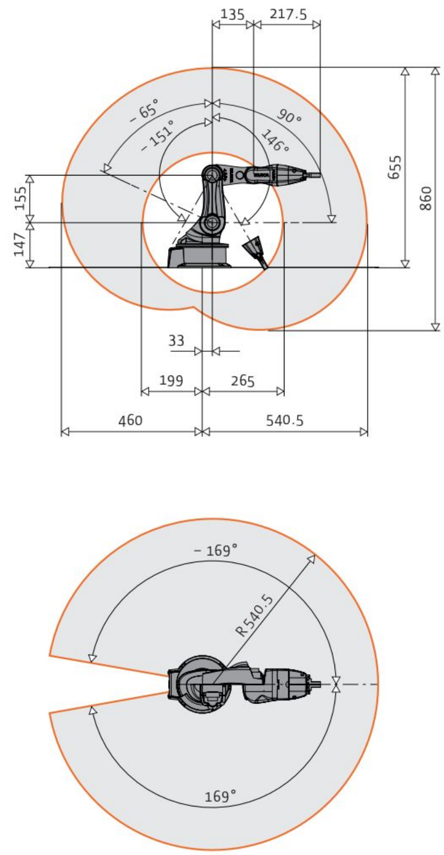

Figure 1: The KUKA youBot Arm workspace. [3]

The dynamic equations of motion have been included in Appendix B. These equations - denoted as α, M, C, and N in the negative null formulation - have been derived using the kinematic model of the robot. They have been generated programmatically using a MATLAB script (specifically `matlabFunction()`) and are fed directly into the optimization algorithm as function handles.

Continuity, Smoothness, Convexity, and Undefined Regions

The objective function and all of the constraints are continuous and smooth on the feasible domain. However, the constraints depending upon α are not continuous and smooth outside the feasible domain. As such there should not be issues like gradient discontinuities when starting within the feasible domain, but would likely be such issues at starting points outside the feasible domain. This is further supported by the fact that no issues were noted when running the solver with our input parameters, suggesting that no discontinuities or undefined regions affect the ability of the solver to converge to a local minimum.

The multi-objective function in the problem is nonconvex. Specifically, the part of the objective function that minimizes the torques that the links experience is parabolic (the inner product of the torque vectors). This is convex, but the final output of this function is also a product of the timestep, which is a decision variable itself that has simple bounds. This plausibly renders this part of the objective function nonconvex, or even concave, in some cases. The other part of the objective function which minimizes the total time of the operation is linear, with the same simple bound on the timestep. As a result, this part is convex.

Natural and Practical Constraints

All of the system dynamics are natural constraints because the physical system must obey the continuous-time equations which they are derived from. The lower and upper bounds on all of the decisions variables are also natural constraints. The numeric values of the parameters are based on the manufacturer limitations of the robot.

Practical constraints include:

- The initial and final conditions of the position and velocity variables (Appendix A).
- The position of the payload.

Modelling Assumptions and Decision Variable Bounds

General assumptions include the following:

- There is no friction between joints.
- There is no air resistance.
- The payload is approximated as simply a point for the end-effector to reach, with no mass.
- The gripper fingers of the youBot Arm are ignored.
- Time is not continuous, but is instead discretized into time steps. So, continuous-time variables are held constant for the duration of each time slice.
- The problem is planar, as opposed to three-dimensional. This eliminates several additional joints that are part of the actual robot.

Please refer to Appendix A for a list of numerical upper and lower bounds for decision variables. Any change in the weighting factor β changes the objective function itself and hence the solution of the problem.

Multimodality and Numerical Difficulties

In considering the nature of our problem, it is highly likely that there are multiple local minima. This is primarily due to the fact that there could be several different means of manipulating the joints to achieve the desired end-effector position, a phenomenon known as kinematic redundancy.

One effort to try and ensure that the solver will converge to fewer values is our team's replacement of our original objective function. Previously, our objective function involved taking an absolute value of the inner product of the velocity and the torque for each link. The absolute value function, while convex, has an obvious discontinuity at the origin. In running initial trials, it was noted that many more distinct local minima were found with this formulation than with our current objective function. As a result, our group opted for the finalized version of the objective function described previously.

Scaling

No scaling attempts were made, as the order of all of the parameters and the decision variables were roughly equivalent (i.e. within a range of approximately one order of magnitude).

Optimization Study

To solve this problem we will be utilizing fmincon on MATLAB. We chose fmincon because our problem is a non-convex NLP. As the problem is non-convex, we opted to use randomized starting points in the hope of finding a global minimum for the problem.

Initially, we started out with a random starting point without much importance given to the feasibility of the starting point. The starting point was generated using a linspace between the initial and final desired states of the joint variables. So, the starting point is an array of states corresponding to each joint variable. This methodology has led to very inconsistent results from the solver. To address this inconsistency we generated a polynomial randomised spline from the initial to the final states of the joint variables. Additionally, these splines were generated to satisfy all the constraints on the joint variables.

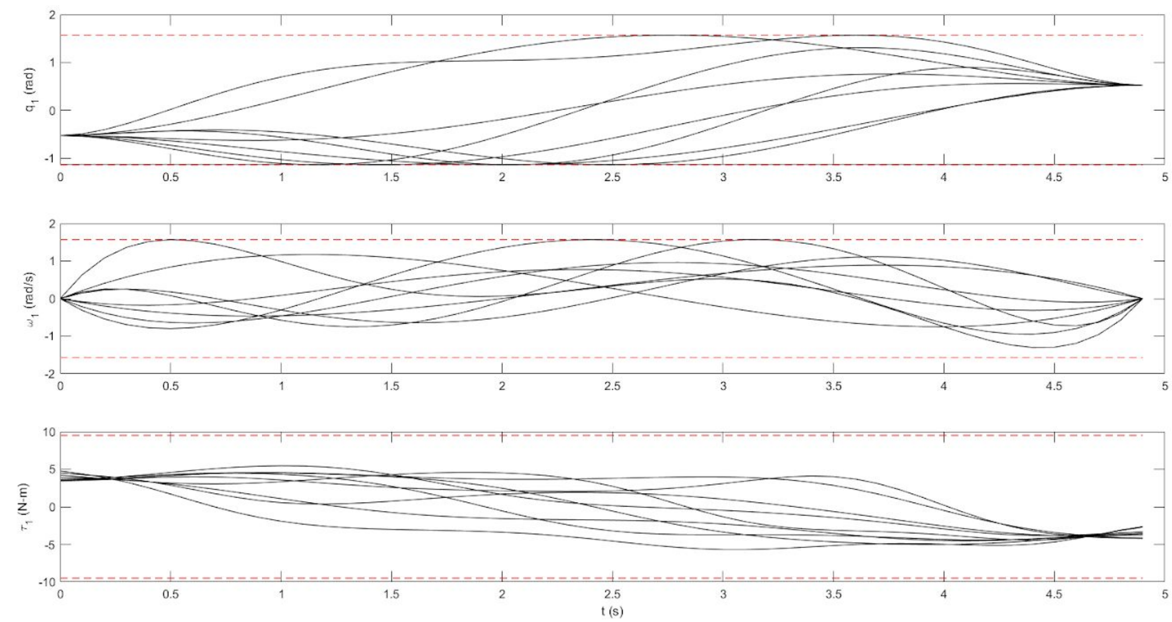

Figure 2. An example set of randomized starting points generated between the initial and final states of joint 1. The three Y - axes (starting from the top) represent the joint angle, speed and torque respectively.

The red dotted lines in Figure 2 represent the constraints on the joint variables. We believe that using these randomized starting points that lie within the feasible domain helped the solver converge to local minima more consistently.

The objective function is a weighted sum of time and work (equivalent) expended. For the work expended section, we will be utilizing the inner-product of the joint torques instead of the absolute values of the work. For the time component, we added in the timestep as part of the input variable to allow fmincon to decide upon the time step based on the fixed number of steps provided by us. For this simulation we used 50 as our number of timesteps. Anything higher often led to a long processing period to solve the problem.

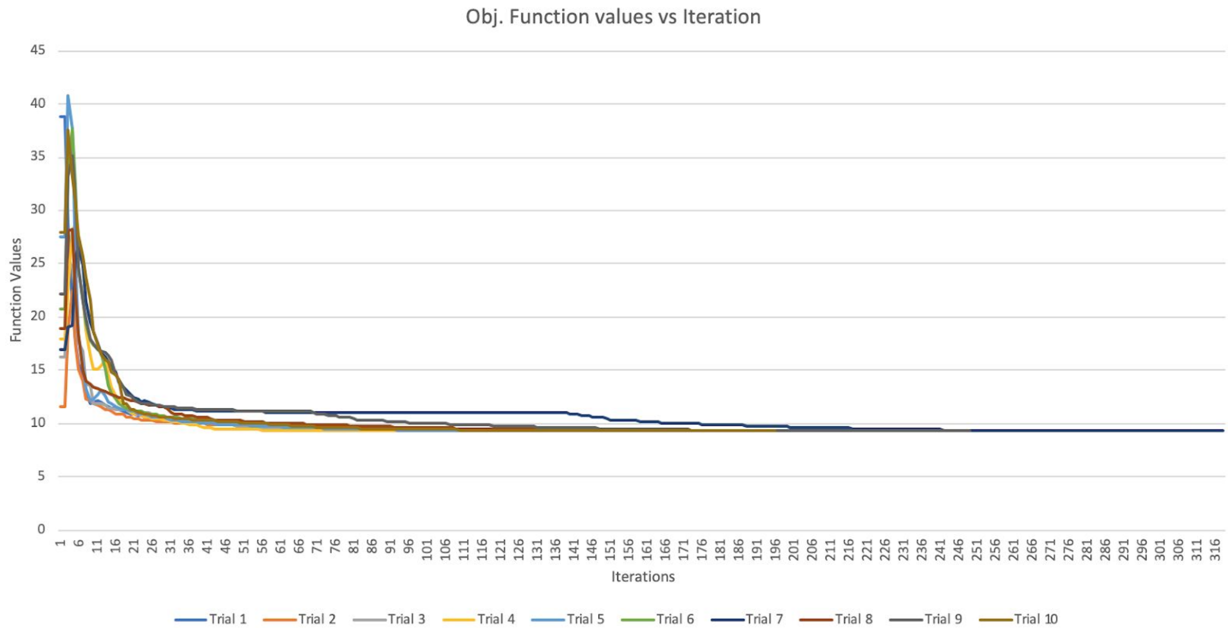

Figure 3. Evolution of the objective value for 10 sample solutions.

In Figure 3, we can see how the iteration data of the function value is changing for the base case of β = 0.5. As we can see, all the points in this 10 randomized starts produce the same result in around 300 iterations. We can also observe the algorithm reaching the region of the final minima, and that it only incrementally decreases for various iterations.

To test the convexity and solvability of the problem under fmincon, we ran the problem for 50 randomized starting points starting from β = 0.1 to 0.9 at a resolution of 0.1 under the following settings:

- Max iterations = 300
- Max function evaluations = ∞ (only max iterations is considered)
- Interior point algorithm

Table 1. Summary of experiments and results for β = 0.1 to 0.9 at a resolution of 0.1.

Criteria	**Information from Solver**	**Interpretation**
Termination	Exit flag = 1 for ~95% of trials (indicating convergence) Exit flag = 0 otherwise (max function evaluations)	Satisfies first order optimality within constraints.
Local optimality	First-order optimality is satisfied (KKT conditions). The BFGS Hessian at the solutions found is positive semidefinite.	Yes, it is a local minima.
Global Optimality	N/A	Used randomized starting points for 50 runs. Suspected global minima with confidence of 50 solver runs.

For 5% of cases where we got an exitflag = 0, it is caused due to fmincon reaching the max function evaluations. So, we tried keeping the max iterations = ∞ and we were able to get an exitflag of 1 in almost all cases.

We further investigated the base case of β = 0.5, with the results summarized in Table 2. The optimal trajectory found is visualized in Figure 4.

Table 2. Summary of experiments and results for β = 0.5.

Criteria	Information from Solver	Interpretation
Termination	Exit flag = 1 for 98% of trials (indicating convergence) Exit flag = 0 otherwise (max function evaluations)	Satisfies first order optimality within constraints.
Local optimality	Two local optima were noted that satisfied first-order optimality: Function evaluation = 9.359 for 92% of trials Function evaluation = 11.070 otherwise	Both are valid local minima.
Global optimality	N/A	Used randomized starting points for 50 runs. Suspected global minima with confidence of 50 solver runs.

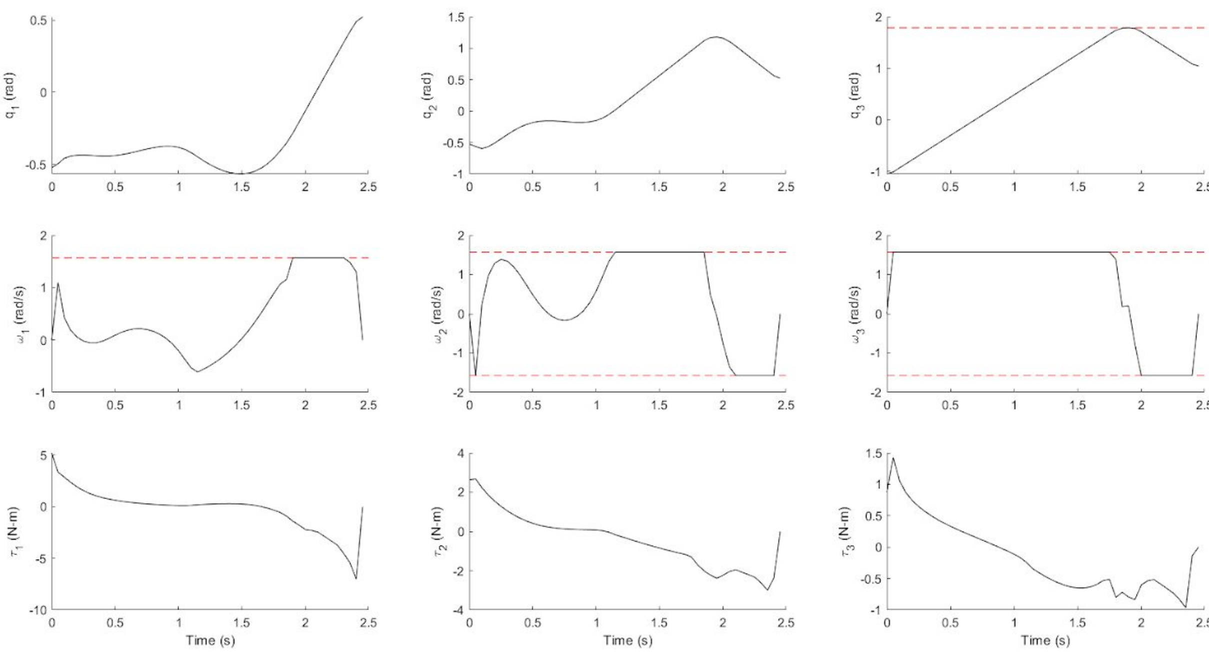

Figure 4. These graphs (top to bottom) show the optimal joint angles, velocities and torques for the base case of β = 0.5

We can see that for mid-ranged values of β, some of the simple bounds on the decision variables, mainly velocity, are active for some time steps of the trajectory. The solution also reduces the initial time step guess of 0.1 s to about 0.050 s. This indicates that the optimization problem is correctly formed to reduce both work expended as well as the total trajectory time span.

Lagrange multipliers

A table of Lagrange multipliers can be found in Appendix C. This table consists of the Lagrange multipliers for the base case of $\beta = 0.5$. As expected, all equality constraints are active. This indicates that the solver is ensuring that the constraints are met. So the initial and final robot poses are being realized, and the collocation is working appropriately. The inequality constraints all represent the simple upper and lower bounds on the decision variables. Figure 4 also shows red dashed lines representing these bounds, and wherever the trajectory overlays these bounds the Lagrange multipliers are also active.

Sensitivity Analysis

The particular type of trajectory optimization being solved here does not have practical constraints in the strictest sense. Of the many individual constraints, they can all be broadly grouped into three categories: collocation constraints, boundary conditions, and joint limits. The first two categories, collocation and boundary constraints, are equality constraints. The collocation constraints are natural since they are discrete-time approximations of continuous-time functions, and are defined by the system's dynamic equations of motion. The boundary constraints are defined by the initial and final joint positions and velocities. Though these values can be changed for different robotic situations, for any particular pick and place operation these are not negotiable.

The last category are the upper and lower bounds on the joint positions, velocities, and torques. For our problem formulation, these are natural constraints since they are the physical limits of the actuators as specified by the manufacturer. However, they could be modified into practical constraints. In practice, it may be preferable to instead define these bounds as some percentage of the true robot joint limits since in a physical implementation the robot would not track the trajectory exactly, so this leaves room for tracking error. Different trajectories can be found depending on how these joint bounds are defined in the problem.

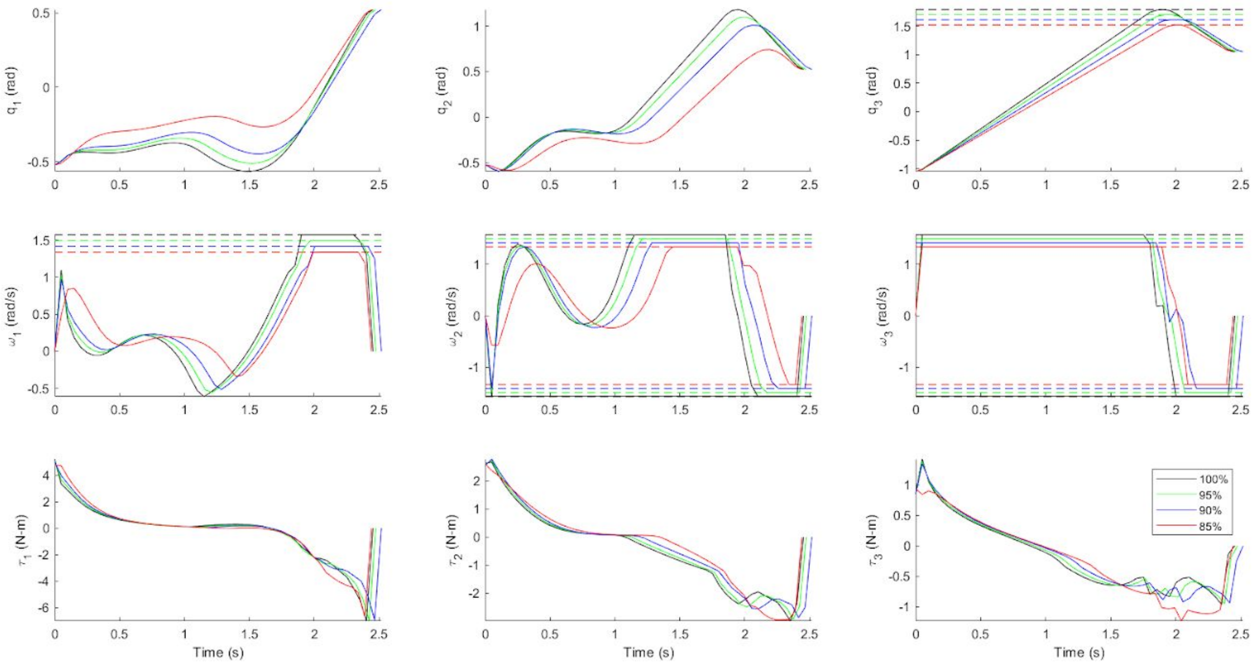

Figure 5. Optimal trajectories for the joint boundaries defined at 100% (black), 95% (green), 90% (blue), and 85% (red) of the manufacturer specifications using β = 0.5. The joint boundaries are shown as dashed lines of corresponding colors for trajectory components which have active inequality constraints.

The above figure shows how the optimal trajectory changes with respect to the joint boundaries. In general, the trajectories all have similar shapes. Additionally, joint limits which were active for the original problem using 100% of the natural limits are also active using the reduced limits. However, there is some unexpected behavior. First, the optimal time does not vary much, and in fact the shortest time of the four samples occurs for the trajectory solved using 85% of the natural limits. Additionally, as the joint limits are constricted, many of them become active for less overall time than the original formulation. The entire trajectory is able to be adapted in order to meet these stricter joint limits in a similar amount of time.

The effects of varying the objective weight can be analyzed in the form of a Pareto frontier. Since we are confident that we can find a global optimum to this problem, the Pareto frontier is able to be generated using sampled solutions.

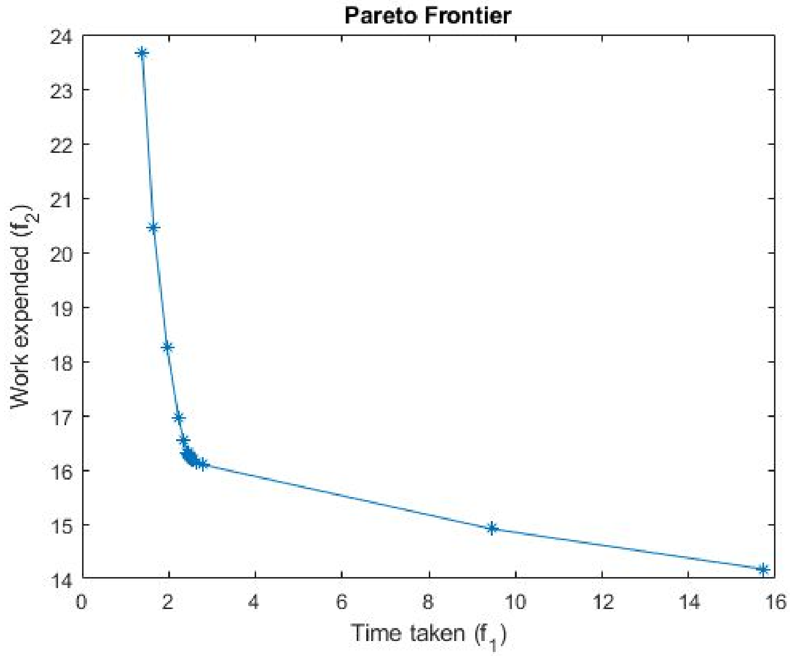

Figure 6. Pareto frontier comparing the tradeoffs between time taken and work expended to execute the optimal trajectory.The blue asterisks represent the global solutions using β from 0.05 (top-left point) to 0.95 (bottom-right point) in increments of 0.05, and are interpolated inbetween.

Figure 6 shows the Pareto frontier for the problem as originally formed. It has several interesting features which have important implications. First, the frontier is not linear, and has a cornering centered around $\beta \approx 0.5$. This implies that the problem has diminishing marginal tradeoffs between time and energy. For example, if a solution is originally found with a large β, then work expended is considered relatively more important than time taken. Using this reference solution, another trajectory generated could take a fraction of the time while only sacrificing a bit more energy if β is reduced. However, as β becomes smaller and smaller much more energy would be needed in order to reduce the time by a marginal amount.

Another interesting feature of the Pareto frontier is that most of the solutions are tightly grouped around the corner. So for a relatively large range of β centered around 0.5, variations do not affect the solution very much. Only as β approaches 0 or 1 does the solution show significant differences.

Conclusions

When considering our problem with the results we obtained, there are several conclusions that our group is able to draw. We noticed that changing the weight parameter β would lead to vastly different trajectories. This was most obvious when selecting a value of β that minimized the torque used across all joints. In running trials for this value, our team noticed that the overall time of the operation was greatly extended. For context, our base case took roughly 2.4 seconds to reach the goal position, while the torque minimization case took 61 seconds. We surmised that the reason for this was that the solver could be exploiting a kinematic singularity in the joints, in addition to lengthening the timestep, to further minimize the torque over time. In contrast, a value of β that minimized the time of the operation - the other extreme - had an operation time of 1.4 seconds.

In looking at how our results could be applied in industry, our group believes that companies who seek to optimize the parts of the objective function in a specific ratio would be able to select a value for the weighting parameter that suits their needs. We predict that most companies would focus on minimizing the time of the operation over the torques, as it is likely that the power required to generate the torques is fairly low, and would therefore provide less cost savings.

References

1. M. Kelly, "An Introduction to Trajectory Optimization: How to Do Your Own Direct Collocation," *Society for Industrial and Applied Mathematics*, vol. 59, no. 4, pp. 849-904, 2017.
2. R. M. Murray et al, "A Mathematical Introduction to Robotic Manipulation." Boca Raton, FL: CRC Press, 1994.
3. "KUKA YouBot Technical Specifications." *KUKA YouBot,* 2015, www.generationrobots.com/img/Kuka-YouBot-Technical-Specs.pdf.

Appendix A - Decision Variables and Parameters

The decision variable vectors relating to position, velocity, and force are composed of the individual scalar components which have physical meanings relating to the robot. The three vectors are expanded below.

$$\mathbf{x}_k^{\mathrm{P}} = [q_k^1 \quad q_k^2 \quad q_k^3]^T \qquad \forall k \in \{1, \dots, n\}$$
$$\mathbf{x}_k^{\mathrm{V}} = [\omega_k^1 \quad \omega_k^2 \quad \omega_k^3]^T \qquad \forall k \in \{1, \dots, n\}$$
$$\mathbf{x}_k^{\mathrm{F}} = [\tau_k^1 \quad \tau_k^2 \quad \tau_k^3]^T \qquad \forall k \in \{1, \dots, n\}$$

For each vector, the elements correspond to joints 1, 2, and 3, respectively. So for the position vector, the first element is for joint 1, the next joint 2, and the last joint 3. The velocity and torque vectors follow an identical pattern.

It is important to distinguish between the joint numbering used in this optimization problem and the joint numbering of the actual Kuka youBot. The youBot robotic arm has five joints, of which only three were modeled for this problem, and the appropriate parameters combined accordingly. So joints 1, 2, and 3 for this problem are actually joints 2, 3, and 4 of the youBot robotic arm, respectively. Joints 1 and 5 of the youBot arm are considered rigid.

Table A1. Model parameter values. [3]

Symbol	Description	Value	Sensitivity Values	Units
n	Number of discrete time points	50	-	-
β	Objective weighting factor	0.5	[0.05, 0.95]	-
$\mathbf{x}_{k,\mathrm{L}}^{\mathrm{P}}$	Position decision variables upper bounds $\forall k \in \{1, \dots, n\}$	-65 -151 -102.5	{-61.75, -58.5, -55.25} {-143.45, -135.9, -128.35} {-97.375, -92.25, -87.125}	° ° °
$\mathbf{x}_{k,\mathrm{U}}^{\mathrm{P}}$	Position decision variables upper bounds $\forall k \in \{1, \dots, n\}$	90 146 102.5	{85.5, 81, 76.5} {138.7, 131.4, 124.1} {97.375, 92.25, 87.125}	° ° °
$\mathbf{x}_{k,\mathrm{L}}^{\mathrm{V}}$	Velocity decision variables lower bounds $\forall k \in \{1, \dots, n\}$	-90 -90 -90	{-85.5, -81, -76.5} {-85.5, -81, -76.5} {-85.5, -81, -76.5}	°/s °/s °/s
$\mathbf{x}_{k,\mathrm{U}}^{\mathrm{V}}$	Velocity decision variables upper bounds $\forall k \in \{1, \dots, n\}$	90 90 90	{85.5, 81, 76.5} {85.5, 81, 76.5} {85.5, 81, 76.5}	°/s °/s °/s
$\mathbf{x}_{k,\mathrm{L}}^{\mathrm{F}}$	Force decision variables lower bounds $\forall k \in \{1, \dots, n\}$	-9.5 -6.0 -2.0	{-9.052, -8.55, -8.075} {-5.7, -5.4, -5.1} {-1.9, -1.8, -1.7}	N-m N-m N-m
$\mathbf{x}_{k,\mathrm{U}}^{\mathrm{F}}$	Force decision variables upper bounds $\forall k \in \{1, \dots, n\}$	9.5 6.0 2.0	{9.052, 8.55, 8.075} {5.7, 5.4, 5.1} {1.9, 1.8, 1.7}	N-m N-m N-m
$\mathbf{x}_{\mathrm{I}}^{\mathrm{P}}$	Position decision variables initial conditions	-π/6 -π/6 -π/3	- - -	rad rad rad
$\mathbf{x}_{\mathrm{F}}^{\mathrm{P}}$	Position decision variables final conditions	π/6 π/6 π/3	- - -	rad rad rad
$\mathbf{x}_{\mathrm{I}}^{\mathrm{V}}$	Velocity decision variables initial conditions	0 0 0	- - -	rad/s rad/s rad/s
$\mathbf{x}_{\mathrm{F}}^{\mathrm{V}}$	Velocity decision variables final conditions	0 0 0	- - -	rad/s rad/s rad/s
$\mathbf{d}_{\mathrm{R}}$	Robotic arm link lengths	155 135 217.5	- - -	mm mm mm
$\boldsymbol{\theta}_{\mathrm{R}}$	Robotic arm link lengths to centers of mass	77.5 67.5 108.75	- - -	mm mm mm
$\mathbf{m}_{\mathrm{R}}^{\mathrm{L}}$	Robotic arm link masses	1.318 0.821 0.769	- - -	kg kg kg
$\mathbf{m}_{\mathrm{R}}^{\mathrm{XX}}$	Robotic arm link moments of inertia about X	0.0031145 0.00172767 0.0006764	- - -	kg-m² kg-m² kg-m²

Appendix B - Dynamic Equations of Motion

One of the most critical components of the problem model is the robotic arm's dynamic equations of motion. These dictate the physics by which the robot moves. It is especially important since it directly relates the joint accelerations to the positions, velocities, and torques. The dynamics of this system are modeled using Lagrangian dynamics. [2] This results in a set of matrices which are functions of the decision variables and parameters. However, these matrix functions are symbolic and much too large and complex to write out directly. Instead, the derivation of these matrices is provided here.

First will be the mass matrix. For each robot link, a link mass matrix is formed which aligns with each respective link's principal inertial axes. Since the principal axes of the links of this robot align with its joint axes, each of these matrices is a simple diagonal.

$$\boldsymbol{M}^i = \begin{bmatrix} m_i^{\mathrm{L}} & 0 & 0 \\ 0 & m_i^{\mathrm{L}} & 0 \\ 0 & 0 & m_i^{\mathrm{XX}} \end{bmatrix} \quad \forall i \in \{1,2,3\}$$

Next, transformation matrices define the kinematic relationships between the links. For this robot, three transformation matrices are ultimately needed: from joint 1 to the centers of mass of each link. The matrices below show this formulation with the intermediate transformation matrices needed to accomplish this. The superscripts represent the starting point followed by the ending point, where S indicates joint 1 and L_i represents the center of mass of link *i*.

$$\boldsymbol{g}_k^{\mathrm{S,L_1}} = \begin{bmatrix} \cos q_k^1 & -\sin q_k^1 & \theta_1 \cos q_k^1 \\ \sin q_k^1 & \cos q_k^1 & \theta_1 \sin q_k^1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{g}_k^{\mathrm{L_1,L_2}} = \begin{bmatrix} \cos q_k^2 & -\sin q_k^2 & d_1 - \theta_1 + \theta_2 \cos q_k^2 \\ \sin q_k^2 & \cos q_k^2 & \theta_2 \sin q_k^2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{g}_k^{\mathrm{L_2,L_3}} = \begin{bmatrix} \cos q_k^3 & -\sin q_k^3 & d_2 - \theta_2 + \theta_3 \cos q_k^3 \\ \sin q_k^3 & \cos q_k^3 & \theta_3 \sin q_k^3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{g}_k^{\mathrm{S,L_2}} = \boldsymbol{g}_k^{\mathrm{S,L_1}} \boldsymbol{g}_k^{\mathrm{L_1,L_2}}$$

$$\boldsymbol{g}_k^{\mathrm{S,L_3}} = \boldsymbol{g}_k^{\mathrm{S,L_1}} \boldsymbol{g}_k^{\mathrm{L_1,L_2}} \boldsymbol{g}_k^{\mathrm{L_2,L_3}}$$

With these transformation matrices defined, the Jacobian matrices of each link can be derived. The Jacobian matrices relate the kinematics defined in the coordinate systems of the links back to the coordinate system of the robot.

$$\boldsymbol{J}_k^{\mathrm{S,L_1}} = \left[\left(\left(\boldsymbol{g}_i^{\mathrm{S,L_1}}\right)^{-1} \frac{\partial}{\partial q_k^1} \boldsymbol{g}_i^{\mathrm{S,L_1}} \right)^{\vee} \quad \boldsymbol{0} \quad \boldsymbol{0} \right]$$

$$\boldsymbol{J}_k^{\mathrm{S,L_2}} = \left[\left(\left(\boldsymbol{g}_k^{\mathrm{S,L_2}}\right)^{-1} \frac{\partial}{\partial q_k^1} \boldsymbol{g}_k^{\mathrm{S,L_2}} \right)^{\vee} \quad \left(\left(\boldsymbol{g}_k^{\mathrm{S,L_2}}\right)^{-1} \frac{\partial}{\partial q_k^2} \boldsymbol{g}_k^{\mathrm{S,L_2}} \right)^{\vee} \quad \boldsymbol{0} \right]$$

$$\boldsymbol{J}_k^{\mathrm{S,L_3}} = \left[\left(\left(\boldsymbol{g}_k^{\mathrm{S,L_3}}\right)^{-1} \frac{\partial}{\partial q_k^1} \boldsymbol{g}_k^{\mathrm{S,L_3}} \right)^{\vee} \quad \left(\left(\boldsymbol{g}_k^{\mathrm{S,L_3}}\right)^{-1} \frac{\partial}{\partial q_k^2} \boldsymbol{g}_k^{\mathrm{S,L_3}} \right)^{\vee} \quad \left(\left(\boldsymbol{g}_k^{\mathrm{S,L_3}}\right)^{-1} \frac{\partial}{\partial q_k^3} \boldsymbol{g}_k^{\mathrm{S,L_3}} \right)^{\vee} \right]$$

The special “vee” operator simply rearranges the resulting matrix elements into a twist coordinate vector form. The operation is shown below to a generic matrix.

$$\boldsymbol{A}^{\vee} = \begin{bmatrix} 0 & -\omega & x \\ \omega & 0 & y \\ 0 & 0 & 0 \end{bmatrix}^{\vee} = \begin{bmatrix} x \\ y \\ \omega \end{bmatrix}$$

Finally, the robot mass matrix is found by summing the individual link mass matrices multiplied by their respective Jacobian matrices squared.

$$\boldsymbol{M}_k = \sum_{i=1}^{3} \left(\boldsymbol{J}_k^{\mathrm{S,L}_i}\right)^T \boldsymbol{M}^i \boldsymbol{J}_k^{\mathrm{S,L}_i} = \left(\boldsymbol{J}_k^{\mathrm{S,L_1}}\right)^T \boldsymbol{M}^1 \boldsymbol{J}_k^{\mathrm{S,L_1}} + \left(\boldsymbol{J}_k^{\mathrm{S,L_2}}\right)^T \boldsymbol{M}^2 \boldsymbol{J}_k^{\mathrm{S,L_2}} + \left(\boldsymbol{J}_k^{\mathrm{S,L_3}}\right)^T \boldsymbol{M}^3 \boldsymbol{J}_k^{\mathrm{S,L_3}} = \begin{bmatrix} M_k^{11} & M_k^{12} & M_k^{13} \\ M_k^{21} & M_k^{22} & M_k^{23} \\ M_k^{31} & M_k^{32} & M_k^{33} \end{bmatrix}$$

The next matrix to derive is the Coriolis matrix. This matrix is of the same size as the mass matrix, and each element has a derivation based on a summation of Christoffel symbols of the first kind for the mass matrix.

$$\boldsymbol{C}_k = \begin{bmatrix} C_k^{11} & C_k^{12} & C_k^{13} \\ C_k^{21} & C_k^{22} & C_k^{23} \\ C_k^{31} & C_k^{32} & C_k^{33} \end{bmatrix}$$

$$C_k^{ij} = \frac{1}{2} \sum_{h=1}^{3} \left(\frac{\partial}{\partial q_k^h} M_k^{ij} + \frac{\partial}{\partial q_k^j} M_k^{ih} - \frac{\partial}{\partial q_k^i} M_k^{hj} \right) \omega_k^h$$

The last matrix to derive for the complete Lagrangian dynamics system of equations is the nonlinear vector. This vector captures all of the nonlinear dynamics in the system. Based on the modeling assumptions made for this problem, the only nonlinear forces present are the forces of gravity on each link. First, the overall robot gravitational potential energy is derived. It is essentially the sum of the individual link gravitational potential energies measured from a common reference, in this case joint 1.

$$V_k = m_1^{\mathrm{L}} g \theta_1 \cos q_k^1 + m_2^{\mathrm{L}} g \left(d_1 \cos q_k^1 + \theta_2 \cos\left(q_k^1 + q_k^2\right)\right) \\ + m_3^{\mathrm{L}} g \left(d_1 \cos q_k^1 + d_2 \cos\left(q_k^1 + q_k^2\right) + \theta_3 \cos\left(q_k^1 + q_k^2 + q_k^3\right)\right)$$

Last, the nonlinear vector is simply the gradient of the potential energy.

$$\boldsymbol{N}_k = \left[\frac{\partial}{\partial q_k^1} V_k \quad \frac{\partial}{\partial q_k^2} V_k \quad \frac{\partial}{\partial q_k^3} V_k \right]^T$$

With the necessary matrix functions for the Lagrangian dynamics derived, the full Lagrangian dynamic equations of motion can be formed.

$$\boldsymbol{M}_k \boldsymbol{\alpha}_k + \boldsymbol{C}_k \boldsymbol{x}_k^{\mathrm{V}} + \boldsymbol{N}_k = \boldsymbol{x}_k^{\mathrm{F}}$$

Appendix C - List of Lagrange Multipliers for Sample Solution

This table contains the constraints for the sample solution generated. For each constraint, it lists the type, Lagrange multiplier, and activity.

Constraint	Type	Time Index	Lagrange Multiplier	Active
q^1 initial condition	Equality	1	5.321307323 rad^{-1}	Yes
q^2 initial condition	Equality	1	2.806134729 rad^{-1}	Yes
q^3 initial condition	Equality	1	0.705209688 rad^{-1}	Yes
ω^1 initial condition	Equality	1	1.591201129 s/rad	Yes
ω^2 initial condition	Equality	1	0.964194543 s/rad	Yes
ω^3 initial condition	Equality	1	0.023662567 s/rad	Yes
q^1 final condition	Equality	50	-7.177642561 rad^{-1}	Yes
q^2 final condition	Equality	50	1.911441733 rad^{-1}	Yes
q^3 final condition	Equality	50	3.319463026 rad^{-1}	Yes
ω^1 final condition	Equality	50	2.271688858 s/rad	Yes
ω^2 final condition	Equality	50	0.981409221 s/rad	Yes
ω^3 final condition	Equality	50	0.187511129 s/rad	Yes
q^1 lower bound	Inequality	1	2.62E-06 rad^{-1}	No
q^2 lower bound	Inequality	1	7.58E-07 rad^{-1}	No
q^3 lower bound	Inequality	1	2.16E-06 rad^{-1}	No
ω^1 lower bound	Inequality	1	1.02E-06 s/rad	No
ω^2 lower bound	Inequality	1	1.02E-06 s/rad	No
ω^3 lower bound	Inequality	1	1.02E-06 s/rad	No
τ^1 lower bound	Inequality	1	1.09E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	1	1.85E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	1	5.57E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	2	2.51E-06 rad^{-1}	No
q^2 lower bound	Inequality	2	7.72E-07 rad^{-1}	No
q^3 lower bound	Inequality	2	2.05E-06 rad^{-1}	No
ω^1 lower bound	Inequality	2	6.00E-07 s/rad	No
ω^2 lower bound	Inequality	2	0.106471614 s/rad	Yes
ω^3 lower bound	Inequality	2	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	2	1.25E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	2	1.84E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	2	4.67E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	3	2.37E-06 rad^{-1}	No
q^2 lower bound	Inequality	3	7.85E-07 rad^{-1}	No
q^3 lower bound	Inequality	3	1.86E-06 rad^{-1}	No
ω^1 lower bound	Inequality	3	8.02E-07 s/rad	No
ω^2 lower bound	Inequality	3	8.80E-07 s/rad	No
ω^3 lower bound	Inequality	3	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	3	1.29E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	3	1.95E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	3	5.24E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	4	2.31E-06 rad^{-1}	No
q^2 lower bound	Inequality	4	7.73E-07 rad^{-1}	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
q^3 lower bound	Inequality	4	1.71E-06 rad^{-1}	No
ω^1 lower bound	Inequality	4	9.13E-07 s/rad	No
ω^2 lower bound	Inequality	4	6.29E-07 s/rad	No
ω^3 lower bound	Inequality	4	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	4	1.35E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	4	2.04E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	4	5.58E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	5	2.29E-06 rad^{-1}	No
q^2 lower bound	Inequality	5	7.52E-07 rad^{-1}	No
q^3 lower bound	Inequality	5	1.57E-06 rad^{-1}	No
ω^1 lower bound	Inequality	5	9.83E-07 s/rad	No
ω^2 lower bound	Inequality	5	5.60E-07 s/rad	No
ω^3 lower bound	Inequality	5	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	5	1.40E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	5	2.12E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	5	5.83E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	6	2.29E-06 rad^{-1}	No
q^2 lower bound	Inequality	6	7.29E-07 rad^{-1}	No
q^3 lower bound	Inequality	6	1.46E-06 rad^{-1}	No
ω^1 lower bound	Inequality	6	1.03E-06 s/rad	No
ω^2 lower bound	Inequality	6	5.41E-07 s/rad	No
ω^3 lower bound	Inequality	6	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	6	1.45E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	6	2.20E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	6	6.04E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	7	2.30E-06 rad^{-1}	No
q^2 lower bound	Inequality	7	7.07E-07 rad^{-1}	No
q^3 lower bound	Inequality	7	1.36E-06 rad^{-1}	No
ω^1 lower bound	Inequality	7	1.06E-06 s/rad	No
ω^2 lower bound	Inequality	7	5.49E-07 s/rad	No
ω^3 lower bound	Inequality	7	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	7	1.48E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	7	2.27E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	7	6.23E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	8	2.31E-06 rad^{-1}	No
q^2 lower bound	Inequality	8	6.88E-07 rad^{-1}	No
q^3 lower bound	Inequality	8	1.28E-06 rad^{-1}	No
ω^1 lower bound	Inequality	8	1.06E-06 s/rad	No
ω^2 lower bound	Inequality	8	5.77E-07 s/rad	No
ω^3 lower bound	Inequality	8	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	8	1.52E-07 $(N\text{-}m)^{-1}$	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
τ^2 lower bound	Inequality	8	2.34E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	8	6.40E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	9	2.31E-06 rad^{-1}	No
q^2 lower bound	Inequality	9	6.72E-07 rad^{-1}	No
q^3 lower bound	Inequality	9	1.20E-06 rad^{-1}	No
ω^1 lower bound	Inequality	9	1.04E-06 s/rad	No
ω^2 lower bound	Inequality	9	6.25E-07 s/rad	No
ω^3 lower bound	Inequality	9	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	9	1.54E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	9	2.40E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	9	6.57E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	10	2.31E-06 rad^{-1}	No
q^2 lower bound	Inequality	10	6.60E-07 rad^{-1}	No
q^3 lower bound	Inequality	10	1.13E-06 rad^{-1}	No
ω^1 lower bound	Inequality	10	1.00E-06 s/rad	No
ω^2 lower bound	Inequality	10	6.93E-07 s/rad	No
ω^3 lower bound	Inequality	10	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	10	1.56E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	10	2.45E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	10	6.72E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	11	2.30E-06 rad^{-1}	No
q^2 lower bound	Inequality	11	6.52E-07 rad^{-1}	No
q^3 lower bound	Inequality	11	1.07E-06 rad^{-1}	No
ω^1 lower bound	Inequality	11	9.64E-07 s/rad	No
ω^2 lower bound	Inequality	11	7.77E-07 s/rad	No
ω^3 lower bound	Inequality	11	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	11	1.58E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	11	2.50E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	11	6.86E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	12	2.28E-06 rad^{-1}	No
q^2 lower bound	Inequality	12	6.47E-07 rad^{-1}	No
q^3 lower bound	Inequality	12	1.02E-06 rad^{-1}	No
ω^1 lower bound	Inequality	12	9.31E-07 s/rad	No
ω^2 lower bound	Inequality	12	8.75E-07 s/rad	No
ω^3 lower bound	Inequality	12	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	12	1.59E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	12	2.53E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	12	7.00E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	13	2.26E-06 rad^{-1}	No
q^2 lower bound	Inequality	13	6.45E-07 rad^{-1}	No
q^3 lower bound	Inequality	13	9.72E-07 rad^{-1}	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
ω^1 lower bound	Inequality	13	9.09E-07 s/rad	No
ω^2 lower bound	Inequality	13	9.74E-07 s/rad	No
ω^3 lower bound	Inequality	13	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	13	1.61E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	13	2.56E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	13	7.14E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	14	2.23E-06 rad^{-1}	No
q^2 lower bound	Inequality	14	6.45E-07 rad^{-1}	No
q^3 lower bound	Inequality	14	9.28E-07 rad^{-1}	No
ω^1 lower bound	Inequality	14	8.98E-07 s/rad	No
ω^2 lower bound	Inequality	14	1.06E-06 s/rad	No
ω^3 lower bound	Inequality	14	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	14	1.62E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	14	2.58E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	14	7.28E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	15	2.19E-06 rad^{-1}	No
q^2 lower bound	Inequality	15	6.46E-07 rad^{-1}	No
q^3 lower bound	Inequality	15	8.87E-07 rad^{-1}	No
ω^1 lower bound	Inequality	15	8.95E-07 s/rad	No
ω^2 lower bound	Inequality	15	1.12E-06 s/rad	No
ω^3 lower bound	Inequality	15	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	15	1.63E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	15	2.60E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	15	7.43E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	16	2.16E-06 rad^{-1}	No
q^2 lower bound	Inequality	16	6.48E-07 rad^{-1}	No
q^3 lower bound	Inequality	16	8.50E-07 rad^{-1}	No
ω^1 lower bound	Inequality	16	9.03E-07 s/rad	No
ω^2 lower bound	Inequality	16	1.14E-06 s/rad	No
ω^3 lower bound	Inequality	16	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	16	1.64E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	16	2.61E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	16	7.58E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	17	2.14E-06 rad^{-1}	No
q^2 lower bound	Inequality	17	6.50E-07 rad^{-1}	No
q^3 lower bound	Inequality	17	8.16E-07 rad^{-1}	No
ω^1 lower bound	Inequality	17	9.20E-07 s/rad	No
ω^2 lower bound	Inequality	17	1.12E-06 s/rad	No
ω^3 lower bound	Inequality	17	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	17	1.64E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	17	2.62E-07 $(N\text{-}m)^{-1}$	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
τ^3 lower bound	Inequality	17	7.73E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	18	2.12E-06 rad^{-1}	No
q^2 lower bound	Inequality	18	6.52E-07 rad^{-1}	No
q^3 lower bound	Inequality	18	7.85E-07 rad^{-1}	No
ω^1 lower bound	Inequality	18	9.50E-07 s/rad	No
ω^2 lower bound	Inequality	18	1.06E-06 s/rad	No
ω^3 lower bound	Inequality	18	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	18	1.65E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	18	2.62E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	18	7.90E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	19	2.10E-06 rad^{-1}	No
q^2 lower bound	Inequality	19	6.52E-07 rad^{-1}	No
q^3 lower bound	Inequality	19	7.55E-07 rad^{-1}	No
ω^1 lower bound	Inequality	19	9.96E-07 s/rad	No
ω^2 lower bound	Inequality	19	9.73E-07 s/rad	No
ω^3 lower bound	Inequality	19	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	19	1.66E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	19	2.63E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	19	8.07E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	20	2.11E-06 rad^{-1}	No
q^2 lower bound	Inequality	20	6.49E-07 rad^{-1}	No
q^3 lower bound	Inequality	20	7.28E-07 rad^{-1}	No
ω^1 lower bound	Inequality	20	1.07E-06 s/rad	No
ω^2 lower bound	Inequality	20	8.59E-07 s/rad	No
ω^3 lower bound	Inequality	20	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	20	1.66E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	20	2.63E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	20	8.27E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	21	2.13E-06 rad^{-1}	No
q^2 lower bound	Inequality	21	6.44E-07 rad^{-1}	No
q^3 lower bound	Inequality	21	7.03E-07 rad^{-1}	No
ω^1 lower bound	Inequality	21	1.17E-06 s/rad	No
ω^2 lower bound	Inequality	21	7.45E-07 s/rad	No
ω^3 lower bound	Inequality	21	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	21	1.66E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	21	2.64E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	21	8.48E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	22	2.17E-06 rad^{-1}	No
q^2 lower bound	Inequality	22	6.34E-07 rad^{-1}	No
q^3 lower bound	Inequality	22	6.80E-07 rad^{-1}	No
ω^1 lower bound	Inequality	22	1.33E-06 s/rad	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
ω^2 lower bound	Inequality	22	6.38E-07 s/rad	No
ω^3 lower bound	Inequality	22	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	22	1.67E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	22	2.65E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	22	8.73E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	23	2.24E-06 rad^{-1}	No
q^2 lower bound	Inequality	23	6.20E-07 rad^{-1}	No
q^3 lower bound	Inequality	23	6.58E-07 rad^{-1}	No
ω^1 lower bound	Inequality	23	1.55E-06 s/rad	No
ω^2 lower bound	Inequality	23	5.51E-07 s/rad	No
ω^3 lower bound	Inequality	23	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	23	1.66E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	23	2.68E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	23	9.14E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	24	2.33E-06 rad^{-1}	No
q^2 lower bound	Inequality	24	6.03E-07 rad^{-1}	No
q^3 lower bound	Inequality	24	6.37E-07 rad^{-1}	No
ω^1 lower bound	Inequality	24	1.67E-06 s/rad	No
ω^2 lower bound	Inequality	24	5.10E-07 s/rad	No
ω^3 lower bound	Inequality	24	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	24	1.65E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	24	2.74E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	24	9.68E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	25	2.43E-06 rad^{-1}	No
q^2 lower bound	Inequality	25	5.86E-07 rad^{-1}	No
q^3 lower bound	Inequality	25	6.18E-07 rad^{-1}	No
ω^1 lower bound	Inequality	25	1.57E-06 s/rad	No
ω^2 lower bound	Inequality	25	5.09E-07 s/rad	No
ω^3 lower bound	Inequality	25	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	25	1.65E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	25	2.79E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	25	1.01E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	26	2.53E-06 rad^{-1}	No
q^2 lower bound	Inequality	26	5.69E-07 rad^{-1}	No
q^3 lower bound	Inequality	26	6.00E-07 rad^{-1}	No
ω^1 lower bound	Inequality	26	1.48E-06 s/rad	No
ω^2 lower bound	Inequality	26	5.09E-07 s/rad	No
ω^3 lower bound	Inequality	26	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	26	1.64E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	26	2.84E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	26	1.05E-06 $(N\text{-}m)^{-1}$	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
q^1 lower bound	Inequality	27	2.63E-06 rad^{-1}	No
q^2 lower bound	Inequality	27	5.54E-07 rad^{-1}	No
q^3 lower bound	Inequality	27	5.82E-07 rad^{-1}	No
ω^1 lower bound	Inequality	27	1.38E-06 s/rad	No
ω^2 lower bound	Inequality	27	5.09E-07 s/rad	No
ω^3 lower bound	Inequality	27	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	27	1.64E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	27	2.90E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	27	1.08E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	28	2.71E-06 rad^{-1}	No
q^2 lower bound	Inequality	28	5.39E-07 rad^{-1}	No
q^3 lower bound	Inequality	28	5.66E-07 rad^{-1}	No
ω^1 lower bound	Inequality	28	1.28E-06 s/rad	No
ω^2 lower bound	Inequality	28	5.09E-07 s/rad	No
ω^3 lower bound	Inequality	28	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	28	1.63E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	28	2.95E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	28	1.12E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	29	2.77E-06 rad^{-1}	No
q^2 lower bound	Inequality	29	5.25E-07 rad^{-1}	No
q^3 lower bound	Inequality	29	5.51E-07 rad^{-1}	No
ω^1 lower bound	Inequality	29	1.18E-06 s/rad	No
ω^2 lower bound	Inequality	29	5.09E-07 s/rad	No
ω^3 lower bound	Inequality	29	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	29	1.63E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	29	3.00E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	29	1.15E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	30	2.81E-06 rad^{-1}	No
q^2 lower bound	Inequality	30	5.12E-07 rad^{-1}	No
q^3 lower bound	Inequality	30	5.36E-07 rad^{-1}	No
ω^1 lower bound	Inequality	30	1.09E-06 s/rad	No
ω^2 lower bound	Inequality	30	5.09E-07 s/rad	No
ω^3 lower bound	Inequality	30	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	30	1.64E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	30	3.05E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	30	1.17E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	31	2.82E-06 rad^{-1}	No
q^2 lower bound	Inequality	31	4.99E-07 rad^{-1}	No
q^3 lower bound	Inequality	31	5.23E-07 rad^{-1}	No
ω^1 lower bound	Inequality	31	9.99E-07 s/rad	No
ω^2 lower bound	Inequality	31	5.09E-07 s/rad	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
ω^3 lower bound	Inequality	31	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	31	1.64E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	31	3.11E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	31	1.19E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	32	2.79E-06 rad^{-1}	No
q^2 lower bound	Inequality	32	4.87E-07 rad^{-1}	No
q^3 lower bound	Inequality	32	5.10E-07 rad^{-1}	No
ω^1 lower bound	Inequality	32	9.15E-07 s/rad	No
ω^2 lower bound	Inequality	32	5.09E-07 s/rad	No
ω^3 lower bound	Inequality	32	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	32	1.65E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	32	3.16E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	32	1.19E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	33	2.73E-06 rad^{-1}	No
q^2 lower bound	Inequality	33	4.76E-07 rad^{-1}	No
q^3 lower bound	Inequality	33	4.97E-07 rad^{-1}	No
ω^1 lower bound	Inequality	33	8.38E-07 s/rad	No
ω^2 lower bound	Inequality	33	5.09E-07 s/rad	No
ω^3 lower bound	Inequality	33	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	33	1.66E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	33	3.21E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	33	1.17E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	34	2.64E-06 rad^{-1}	No
q^2 lower bound	Inequality	34	4.65E-07 rad^{-1}	No
q^3 lower bound	Inequality	34	4.85E-07 rad^{-1}	No
ω^1 lower bound	Inequality	34	7.67E-07 s/rad	No
ω^2 lower bound	Inequality	34	5.09E-07 s/rad	No
ω^3 lower bound	Inequality	34	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	34	1.68E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	34	3.26E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	34	1.14E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	35	2.51E-06 rad^{-1}	No
q^2 lower bound	Inequality	35	4.55E-07 rad^{-1}	No
q^3 lower bound	Inequality	35	4.74E-07 rad^{-1}	No
ω^1 lower bound	Inequality	35	7.04E-07 s/rad	No
ω^2 lower bound	Inequality	35	5.09E-07 s/rad	No
ω^3 lower bound	Inequality	35	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	35	1.71E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	35	3.31E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	35	1.09E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	36	2.36E-06 rad^{-1}	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
q^2 lower bound	Inequality	36	4.45E-07 rad^{-1}	No
q^3 lower bound	Inequality	36	4.63E-07 rad^{-1}	No
ω^1 lower bound	Inequality	36	6.50E-07 s/rad	No
ω^2 lower bound	Inequality	36	5.09E-07 s/rad	No
ω^3 lower bound	Inequality	36	5.09E-07 s/rad	No
τ^1 lower bound	Inequality	36	1.75E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	36	3.40E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	36	1.08E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	37	2.20E-06 rad^{-1}	No
q^2 lower bound	Inequality	37	4.35E-07 rad^{-1}	No
q^3 lower bound	Inequality	37	4.53E-07 rad^{-1}	No
ω^1 lower bound	Inequality	37	6.07E-07 s/rad	No
ω^2 lower bound	Inequality	37	5.09E-07 s/rad	No
ω^3 lower bound	Inequality	37	5.41E-07 s/rad	No
τ^1 lower bound	Inequality	37	1.79E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	37	3.73E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	37	1.33E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	38	2.05E-06 rad^{-1}	No
q^2 lower bound	Inequality	38	4.26E-07 rad^{-1}	No
q^3 lower bound	Inequality	38	4.48E-07 rad^{-1}	No
ω^1 lower bound	Inequality	38	5.87E-07 s/rad	No
ω^2 lower bound	Inequality	38	5.09E-07 s/rad	No
ω^3 lower bound	Inequality	38	9.11E-07 s/rad	No
τ^1 lower bound	Inequality	38	1.86E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	38	4.01E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	38	1.25E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	39	1.88E-06 rad^{-1}	No
q^2 lower bound	Inequality	39	4.20E-07 rad^{-1}	No
q^3 lower bound	Inequality	39	4.47E-07 rad^{-1}	No
ω^1 lower bound	Inequality	39	5.09E-07 s/rad	No
ω^2 lower bound	Inequality	39	7.78E-07 s/rad	No
ω^3 lower bound	Inequality	39	9.03E-07 s/rad	No
τ^1 lower bound	Inequality	39	1.98E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	39	4.24E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	39	1.33E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	40	1.72E-06 rad^{-1}	No
q^2 lower bound	Inequality	40	4.19E-07 rad^{-1}	No
q^3 lower bound	Inequality	40	4.49E-07 rad^{-1}	No
ω^1 lower bound	Inequality	40	5.09E-07 s/rad	No
ω^2 lower bound	Inequality	40	1.06E-06 s/rad	No
ω^3 lower bound	Inequality	40	2.02E-06 s/rad	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
τ^1 lower bound	Inequality	40	2.07E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	40	4.43E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	40	1.37E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	41	1.59E-06 rad^{-1}	No
q^2 lower bound	Inequality	41	4.21E-07 rad^{-1}	No
q^3 lower bound	Inequality	41	4.57E-07 rad^{-1}	No
ω^1 lower bound	Inequality	41	5.09E-07 s/rad	No
ω^2 lower bound	Inequality	41	1.97E-06 s/rad	No
ω^3 lower bound	Inequality	41	0.00892304 s/rad	Yes
τ^1 lower bound	Inequality	41	2.20E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	41	4.25E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	41	1.15E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	42	1.47E-06 rad^{-1}	No
q^2 lower bound	Inequality	42	4.27E-07 rad^{-1}	No
q^3 lower bound	Inequality	42	4.67E-07 rad^{-1}	No
ω^1 lower bound	Inequality	42	5.09E-07 s/rad	No
ω^2 lower bound	Inequality	42	7.30E-06 s/rad	No
ω^3 lower bound	Inequality	42	0.032006077 s/rad	Yes
τ^1 lower bound	Inequality	42	2.22E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	42	4.05E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	42	1.09E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	43	1.37E-06 rad^{-1}	No
q^2 lower bound	Inequality	43	4.36E-07 rad^{-1}	No
q^3 lower bound	Inequality	43	4.78E-07 rad^{-1}	No
ω^1 lower bound	Inequality	43	5.09E-07 s/rad	No
ω^2 lower bound	Inequality	43	0.021809074 s/rad	Yes
ω^3 lower bound	Inequality	43	0.047556262 s/rad	Yes
τ^1 lower bound	Inequality	43	2.28E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	43	3.95E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	43	1.08E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	44	1.29E-06 rad^{-1}	No
q^2 lower bound	Inequality	44	4.45E-07 rad^{-1}	No
q^3 lower bound	Inequality	44	4.89E-07 rad^{-1}	No
ω^1 lower bound	Inequality	44	5.09E-07 s/rad	No
ω^2 lower bound	Inequality	44	0.063251283 s/rad	Yes
ω^3 lower bound	Inequality	44	0.068227861 s/rad	Yes
τ^1 lower bound	Inequality	44	2.42E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	44	4.08E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	44	1.13E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	45	1.21E-06 rad^{-1}	No
q^2 lower bound	Inequality	45	4.55E-07 rad^{-1}	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
q^3 lower bound	Inequality	45	5.02E-07 rad^{-1}	No
ω^1 lower bound	Inequality	45	5.09E-07 s/rad	No
ω^2 lower bound	Inequality	45	0.079229315 s/rad	Yes
ω^3 lower bound	Inequality	45	0.086792011 s/rad	Yes
τ^1 lower bound	Inequality	45	2.58E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	45	4.22E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	45	1.19E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	46	1.14E-06 rad^{-1}	No
q^2 lower bound	Inequality	46	4.66E-07 rad^{-1}	No
q^3 lower bound	Inequality	46	5.14E-07 rad^{-1}	No
ω^1 lower bound	Inequality	46	5.09E-07 s/rad	No
ω^2 lower bound	Inequality	46	0.09565745 s/rad	Yes
ω^3 lower bound	Inequality	46	0.107378319 s/rad	Yes
τ^1 lower bound	Inequality	46	2.76E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	46	4.36E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	46	1.26E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	47	1.08E-06 rad^{-1}	No
q^2 lower bound	Inequality	47	4.77E-07 rad^{-1}	No
q^3 lower bound	Inequality	47	5.28E-07 rad^{-1}	No
ω^1 lower bound	Inequality	47	5.09E-07 s/rad	No
ω^2 lower bound	Inequality	47	0.164251879 s/rad	Yes
ω^3 lower bound	Inequality	47	0.141802391 s/rad	Yes
τ^1 lower bound	Inequality	47	3.21E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	47	4.75E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	47	1.37E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	48	1.03E-06 rad^{-1}	No
q^2 lower bound	Inequality	48	4.88E-07 rad^{-1}	No
q^3 lower bound	Inequality	48	5.42E-07 rad^{-1}	No
ω^1 lower bound	Inequality	48	5.24E-07 s/rad	No
ω^2 lower bound	Inequality	48	0.207775064 s/rad	Yes
ω^3 lower bound	Inequality	48	0.178294185 s/rad	Yes
τ^1 lower bound	Inequality	48	3.93E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	48	5.32E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	48	1.55E-06 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	49	9.84E-07 rad^{-1}	No
q^2 lower bound	Inequality	49	5.00E-07 rad^{-1}	No
q^3 lower bound	Inequality	49	5.56E-07 rad^{-1}	No
ω^1 lower bound	Inequality	49	5.57E-07 s/rad	No
ω^2 lower bound	Inequality	49	0.21628504 s/rad	Yes
ω^3 lower bound	Inequality	49	0.174518233 s/rad	Yes
τ^1 lower bound	Inequality	49	6.42E-07 $(N\text{-}m)^{-1}$	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
τ^2 lower bound	Inequality	49	4.42E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	49	8.59E-07 $(N\text{-}m)^{-1}$	No
q^1 lower bound	Inequality	50	9.65E-07 rad^{-1}	No
q^2 lower bound	Inequality	50	5.06E-07 rad^{-1}	No
q^3 lower bound	Inequality	50	5.64E-07 rad^{-1}	No
ω^1 lower bound	Inequality	50	1.02E-06 s/rad	No
ω^2 lower bound	Inequality	50	1.02E-06 s/rad	No
ω^3 lower bound	Inequality	50	1.02E-06 s/rad	No
τ^1 lower bound	Inequality	50	1.68E-07 $(N\text{-}m)^{-1}$	No
τ^2 lower bound	Inequality	50	2.67E-07 $(N\text{-}m)^{-1}$	No
τ^3 lower bound	Inequality	50	8.00E-07 $(N\text{-}m)^{-1}$	No
Δt lower bound	Inequality	-	3.20E-05 s^{-1}	No
q^1 upper bound	Inequality	1	7.64E-07 rad^{-1}	No
q^2 upper bound	Inequality	1	5.21E-07 rad^{-1}	No
q^3 upper bound	Inequality	1	5.64E-07 rad^{-1}	No
ω^1 upper bound	Inequality	1	1.02E-06 s/rad	No
ω^2 upper bound	Inequality	1	1.02E-06 s/rad	No
ω^3 upper bound	Inequality	1	1.02E-06 s/rad	No
τ^1 upper bound	Inequality	1	3.73E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	1	4.75E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	1	1.42E-06 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	2	7.74E-07 rad^{-1}	No
q^2 upper bound	Inequality	2	5.14E-07 rad^{-1}	No
q^3 upper bound	Inequality	2	5.72E-07 rad^{-1}	No
ω^1 upper bound	Inequality	2	3.36E-06 s/rad	No
ω^2 upper bound	Inequality	2	5.09E-07 s/rad	No
ω^3 upper bound	Inequality	2	0.248449985 s/rad	Yes
τ^1 upper bound	Inequality	2	2.60E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	2	4.81E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	2	2.79E-06 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	3	7.89E-07 rad^{-1}	No
q^2 upper bound	Inequality	3	5.09E-07 rad^{-1}	No
q^3 upper bound	Inequality	3	5.89E-07 rad^{-1}	No
ω^1 upper bound	Inequality	3	1.39E-06 s/rad	No
ω^2 upper bound	Inequality	3	1.21E-06 s/rad	No
ω^3 upper bound	Inequality	3	0.008190622 s/rad	Yes
τ^1 upper bound	Inequality	3	2.41E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	3	4.24E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	3	1.69E-06 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	4	7.94E-07 rad^{-1}	No
q^2 upper bound	Inequality	4	5.14E-07 rad^{-1}	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
q^{3} upper bound	Inequality	4	6.06E-07 rad^{-1}	No
ω^{1} upper bound	Inequality	4	1.15E-06 s/rad	No
ω^{2} upper bound	Inequality	4	2.67E-06 s/rad	No
ω^{3} upper bound	Inequality	4	0.014806384 s/rad	Yes
τ^{1} upper bound	Inequality	4	2.24E-07 $(N\text{-}m)^{-1}$	No
τ^{2} upper bound	Inequality	4	3.86E-07 $(N\text{-}m)^{-1}$	No
τ^{3} upper bound	Inequality	4	1.41E-06 $(N\text{-}m)^{-1}$	No
q^{1} upper bound	Inequality	5	7.97E-07 rad^{-1}	No
q^{2} upper bound	Inequality	5	5.23E-07 rad^{-1}	No
q^{3} upper bound	Inequality	5	6.25E-07 rad^{-1}	No
ω^{1} upper bound	Inequality	5	1.06E-06 s/rad	No
ω^{2} upper bound	Inequality	5	5.64E-06 s/rad	No
ω^{3} upper bound	Inequality	5	0.016713633 s/rad	Yes
τ^{1} upper bound	Inequality	5	2.11E-07 $(N\text{-}m)^{-1}$	No
τ^{2} upper bound	Inequality	5	3.59E-07 $(N\text{-}m)^{-1}$	No
τ^{3} upper bound	Inequality	5	1.27E-06 $(N\text{-}m)^{-1}$	No
q^{1} upper bound	Inequality	6	7.97E-07 rad^{-1}	No
q^{2} upper bound	Inequality	6	5.35E-07 rad^{-1}	No
q^{3} upper bound	Inequality	6	6.45E-07 rad^{-1}	No
ω^{1} upper bound	Inequality	6	1.01E-06 s/rad	No
ω^{2} upper bound	Inequality	6	8.77E-06 s/rad	No
ω^{3} upper bound	Inequality	6	0.016950531 s/rad	Yes
τ^{1} upper bound	Inequality	6	2.02E-07 $(N\text{-}m)^{-1}$	No
τ^{2} upper bound	Inequality	6	3.39E-07 $(N\text{-}m)^{-1}$	No
τ^{3} upper bound	Inequality	6	1.18E-06 $(N\text{-}m)^{-1}$	No
q^{1} upper bound	Inequality	7	7.96E-07 rad^{-1}	No
q^{2} upper bound	Inequality	7	5.48E-07 rad^{-1}	No
q^{3} upper bound	Inequality	7	6.66E-07 rad^{-1}	No
ω^{1} upper bound	Inequality	7	9.83E-07 s/rad	No
ω^{2} upper bound	Inequality	7	7.10E-06 s/rad	No
ω^{3} upper bound	Inequality	7	0.016597177 s/rad	Yes
τ^{1} upper bound	Inequality	7	1.95E-07 $(N\text{-}m)^{-1}$	No
τ^{2} upper bound	Inequality	7	3.23E-07 $(N\text{-}m)^{-1}$	No
τ^{3} upper bound	Inequality	7	1.12E-06 $(N\text{-}m)^{-1}$	No
q^{1} upper bound	Inequality	8	7.95E-07 rad^{-1}	No
q^{2} upper bound	Inequality	8	5.60E-07 rad^{-1}	No
q^{3} upper bound	Inequality	8	6.88E-07 rad^{-1}	No
ω^{1} upper bound	Inequality	8	9.83E-07 s/rad	No
ω^{2} upper bound	Inequality	8	4.33E-06 s/rad	No
ω^{3} upper bound	Inequality	8	0.016009634 s/rad	Yes
τ^{1} upper bound	Inequality	8	1.89E-07 $(N\text{-}m)^{-1}$	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
τ^2 upper bound	Inequality	8	3.11E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	8	1.07E-06 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	9	7.95E-07 rad^{-1}	No
q^2 upper bound	Inequality	9	5.71E-07 rad^{-1}	No
q^3 upper bound	Inequality	9	7.12E-07 rad^{-1}	No
ω^1 upper bound	Inequality	9	1.00E-06 s/rad	No
ω^2 upper bound	Inequality	9	2.76E-06 s/rad	No
ω^3 upper bound	Inequality	9	0.015416187 s/rad	Yes
τ^1 upper bound	Inequality	9	1.86E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	9	3.00E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	9	1.02E-06 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	10	7.95E-07 rad^{-1}	No
q^2 upper bound	Inequality	10	5.80E-07 rad^{-1}	No
q^3 upper bound	Inequality	10	7.38E-07 rad^{-1}	No
ω^1 upper bound	Inequality	10	1.04E-06 s/rad	No
ω^2 upper bound	Inequality	10	1.92E-06 s/rad	No
ω^3 upper bound	Inequality	10	0.014979471 s/rad	Yes
τ^1 upper bound	Inequality	10	1.83E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	10	2.93E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	10	9.89E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	11	7.96E-07 rad^{-1}	No
q^2 upper bound	Inequality	11	5.86E-07 rad^{-1}	No
q^3 upper bound	Inequality	11	7.66E-07 rad^{-1}	No
ω^1 upper bound	Inequality	11	1.08E-06 s/rad	No
ω^2 upper bound	Inequality	11	1.48E-06 s/rad	No
ω^3 upper bound	Inequality	11	0.014595188 s/rad	Yes
τ^1 upper bound	Inequality	11	1.80E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	11	2.86E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	11	9.59E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	12	7.98E-07 rad^{-1}	No
q^2 upper bound	Inequality	12	5.90E-07 rad^{-1}	No
q^3 upper bound	Inequality	12	7.96E-07 rad^{-1}	No
ω^1 upper bound	Inequality	12	1.12E-06 s/rad	No
ω^2 upper bound	Inequality	12	1.22E-06 s/rad	No
ω^3 upper bound	Inequality	12	0.014348138 s/rad	Yes
τ^1 upper bound	Inequality	12	1.78E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	12	2.82E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	12	9.33E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	13	8.01E-07 rad^{-1}	No
q^2 upper bound	Inequality	13	5.92E-07 rad^{-1}	No
q^3 upper bound	Inequality	13	8.28E-07 rad^{-1}	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
ω^1 upper bound	Inequality	13	1.16E-06 s/rad	No
ω^2 upper bound	Inequality	13	1.07E-06 s/rad	No
ω^3 upper bound	Inequality	13	0.014179192 s/rad	Yes
τ^1 upper bound	Inequality	13	1.77E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	13	2.78E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	13	9.09E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	14	8.06E-07 rad^{-1}	No
q^2 upper bound	Inequality	14	5.92E-07 rad^{-1}	No
q^3 upper bound	Inequality	14	8.63E-07 rad^{-1}	No
ω^1 upper bound	Inequality	14	1.18E-06 s/rad	No
ω^2 upper bound	Inequality	14	9.81E-07 s/rad	No
ω^3 upper bound	Inequality	14	0.014081894 s/rad	Yes
τ^1 upper bound	Inequality	14	1.75E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	14	2.76E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	14	8.87E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	15	8.10E-07 rad^{-1}	No
q^2 upper bound	Inequality	15	5.91E-07 rad^{-1}	No
q^3 upper bound	Inequality	15	9.02E-07 rad^{-1}	No
ω^1 upper bound	Inequality	15	1.18E-06 s/rad	No
ω^2 upper bound	Inequality	15	9.34E-07 s/rad	No
ω^3 upper bound	Inequality	15	0.014090784 s/rad	Yes
τ^1 upper bound	Inequality	15	1.74E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	15	2.74E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	15	8.67E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	16	8.14E-07 rad^{-1}	No
q^2 upper bound	Inequality	16	5.89E-07 rad^{-1}	No
q^3 upper bound	Inequality	16	9.43E-07 rad^{-1}	No
ω^1 upper bound	Inequality	16	1.17E-06 s/rad	No
ω^2 upper bound	Inequality	16	9.18E-07 s/rad	No
ω^3 upper bound	Inequality	16	0.014165319 s/rad	Yes
τ^1 upper bound	Inequality	16	1.73E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	16	2.72E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	16	8.47E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	17	8.18E-07 rad^{-1}	No
q^2 upper bound	Inequality	17	5.88E-07 rad^{-1}	No
q^3 upper bound	Inequality	17	9.89E-07 rad^{-1}	No
ω^1 upper bound	Inequality	17	1.14E-06 s/rad	No
ω^2 upper bound	Inequality	17	9.31E-07 s/rad	No
ω^3 upper bound	Inequality	17	0.014283792 s/rad	Yes
τ^1 upper bound	Inequality	17	1.73E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	17	2.72E-07 $(N\text{-}m)^{-1}$	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
τ^3 upper bound	Inequality	17	8.29E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	18	8.21E-07 rad^{-1}	No
q^2 upper bound	Inequality	18	5.86E-07 rad^{-1}	No
q^3 upper bound	Inequality	18	1.04E-06 rad^{-1}	No
ω^1 upper bound	Inequality	18	1.10E-06 s/rad	No
ω^2 upper bound	Inequality	18	9.76E-07 s/rad	No
ω^3 upper bound	Inequality	18	0.014458594 s/rad	Yes
τ^1 upper bound	Inequality	18	1.72E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	18	2.71E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	18	8.10E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	19	8.23E-07 rad^{-1}	No
q^2 upper bound	Inequality	19	5.86E-07 rad^{-1}	No
q^3 upper bound	Inequality	19	1.10E-06 rad^{-1}	No
ω^1 upper bound	Inequality	19	1.04E-06 s/rad	No
ω^2 upper bound	Inequality	19	1.07E-06 s/rad	No
ω^3 upper bound	Inequality	19	0.014721767 s/rad	Yes
τ^1 upper bound	Inequality	19	1.71E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	19	2.71E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	19	7.93E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	20	8.22E-07 rad^{-1}	No
q^2 upper bound	Inequality	20	5.88E-07 rad^{-1}	No
q^3 upper bound	Inequality	20	1.16E-06 rad^{-1}	No
ω^1 upper bound	Inequality	20	9.73E-07 s/rad	No
ω^2 upper bound	Inequality	20	1.25E-06 s/rad	No
ω^3 upper bound	Inequality	20	0.014979837 s/rad	Yes
τ^1 upper bound	Inequality	20	1.71E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	20	2.70E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	20	7.75E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	21	8.19E-07 rad^{-1}	No
q^2 upper bound	Inequality	21	5.93E-07 rad^{-1}	No
q^3 upper bound	Inequality	21	1.23E-06 rad^{-1}	No
ω^1 upper bound	Inequality	21	9.00E-07 s/rad	No
ω^2 upper bound	Inequality	21	1.61E-06 s/rad	No
ω^3 upper bound	Inequality	21	0.015327661 s/rad	Yes
τ^1 upper bound	Inequality	21	1.70E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	21	2.70E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	21	7.57E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	22	8.13E-07 rad^{-1}	No
q^2 upper bound	Inequality	22	6.02E-07 rad^{-1}	No
q^3 upper bound	Inequality	22	1.31E-06 rad^{-1}	No
ω^1 upper bound	Inequality	22	8.25E-07 s/rad	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
ω^2 upper bound	Inequality	22	2.53E-06 s/rad	No
ω^3 upper bound	Inequality	22	0.015651053 s/rad	Yes
τ^1 upper bound	Inequality	22	1.70E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	22	2.68E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	22	7.38E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	23	8.04E-07 rad^{-1}	No
q^2 upper bound	Inequality	23	6.15E-07 rad^{-1}	No
q^3 upper bound	Inequality	23	1.40E-06 rad^{-1}	No
ω^1 upper bound	Inequality	23	7.59E-07 s/rad	No
ω^2 upper bound	Inequality	23	6.75E-06 s/rad	No
ω^3 upper bound	Inequality	23	0.015787428 s/rad	Yes
τ^1 upper bound	Inequality	23	1.71E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	23	2.65E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	23	7.11E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	24	7.93E-07 rad^{-1}	No
q^2 upper bound	Inequality	24	6.32E-07 rad^{-1}	No
q^3 upper bound	Inequality	24	1.50E-06 rad^{-1}	No
ω^1 upper bound	Inequality	24	7.34E-07 s/rad	No
ω^2 upper bound	Inequality	24	0.000609058 s/rad	Yes
ω^3 upper bound	Inequality	24	0.016175359 s/rad	Yes
τ^1 upper bound	Inequality	24	1.72E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	24	2.60E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	24	6.82E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	25	7.81E-07 rad^{-1}	No
q^2 upper bound	Inequality	25	6.53E-07 rad^{-1}	No
q^3 upper bound	Inequality	25	1.62E-06 rad^{-1}	No
ω^1 upper bound	Inequality	25	7.53E-07 s/rad	No
ω^2 upper bound	Inequality	25	0.00258244 s/rad	Yes
ω^3 upper bound	Inequality	25	0.017514201 s/rad	Yes
τ^1 upper bound	Inequality	25	1.72E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	25	2.55E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	25	6.63E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	26	7.71E-07 rad^{-1}	No
q^2 upper bound	Inequality	26	6.74E-07 rad^{-1}	No
q^3 upper bound	Inequality	26	1.76E-06 rad^{-1}	No
ω^1 upper bound	Inequality	26	7.78E-07 s/rad	No
ω^2 upper bound	Inequality	26	0.004261014 s/rad	Yes
ω^3 upper bound	Inequality	26	0.018538127 s/rad	Yes
τ^1 upper bound	Inequality	26	1.73E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	26	2.51E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	26	6.47E-07 $(N\text{-}m)^{-1}$	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
q^1 upper bound	Inequality	27	7.63E-07 rad^{-1}	No
q^2 upper bound	Inequality	27	6.97E-07 rad^{-1}	No
q^3 upper bound	Inequality	27	1.93E-06 rad^{-1}	No
ω^1 upper bound	Inequality	27	8.09E-07 s/rad	No
ω^2 upper bound	Inequality	27	0.006069448 s/rad	Yes
ω^3 upper bound	Inequality	27	0.019582479 s/rad	Yes
τ^1 upper bound	Inequality	27	1.73E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	27	2.47E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	27	6.34E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	28	7.57E-07 rad^{-1}	No
q^2 upper bound	Inequality	28	7.22E-07 rad^{-1}	No
q^3 upper bound	Inequality	28	2.13E-06 rad^{-1}	No
ω^1 upper bound	Inequality	28	8.47E-07 s/rad	No
ω^2 upper bound	Inequality	28	0.008002034 s/rad	Yes
ω^3 upper bound	Inequality	28	0.020610218 s/rad	Yes
τ^1 upper bound	Inequality	28	1.74E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	28	2.43E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	28	6.22E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	29	7.52E-07 rad^{-1}	No
q^2 upper bound	Inequality	29	7.49E-07 rad^{-1}	No
q^3 upper bound	Inequality	29	2.38E-06 rad^{-1}	No
ω^1 upper bound	Inequality	29	8.96E-07 s/rad	No
ω^2 upper bound	Inequality	29	0.009981503 s/rad	Yes
ω^3 upper bound	Inequality	29	0.021563199 s/rad	Yes
τ^1 upper bound	Inequality	29	1.74E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	29	2.40E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	29	6.13E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	30	7.49E-07 rad^{-1}	No
q^2 upper bound	Inequality	30	7.77E-07 rad^{-1}	No
q^3 upper bound	Inequality	30	2.69E-06 rad^{-1}	No
ω^1 upper bound	Inequality	30	9.58E-07 s/rad	No
ω^2 upper bound	Inequality	30	0.011994653 s/rad	Yes
ω^3 upper bound	Inequality	30	0.022372586 s/rad	Yes
τ^1 upper bound	Inequality	30	1.74E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	30	2.37E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	30	6.07E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	31	7.49E-07 rad^{-1}	No
q^2 upper bound	Inequality	31	8.08E-07 rad^{-1}	No
q^3 upper bound	Inequality	31	3.10E-06 rad^{-1}	No
ω^1 upper bound	Inequality	31	1.04E-06 s/rad	No
ω^2 upper bound	Inequality	31	0.013966652 s/rad	Yes

Constraint	Type	Time Index	Lagrange Multiplier	Active
ω^3 upper bound	Inequality	31	0.022961098 s/rad	Yes
τ^1 upper bound	Inequality	31	1.73E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	31	2.34E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	31	6.04E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	32	7.50E-07 rad^{-1}	No
q^2 upper bound	Inequality	32	8.42E-07 rad^{-1}	No
q^3 upper bound	Inequality	32	3.66E-06 rad^{-1}	No
ω^1 upper bound	Inequality	32	1.15E-06 s/rad	No
ω^2 upper bound	Inequality	32	0.015855475 s/rad	Yes
ω^3 upper bound	Inequality	32	0.0232477 s/rad	Yes
τ^1 upper bound	Inequality	32	1.72E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	32	2.31E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	32	6.04E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	33	7.55E-07 rad^{-1}	No
q^2 upper bound	Inequality	33	8.78E-07 rad^{-1}	No
q^3 upper bound	Inequality	33	4.46E-06 rad^{-1}	No
ω^1 upper bound	Inequality	33	1.30E-06 s/rad	No
ω^2 upper bound	Inequality	33	0.017538492 s/rad	Yes
ω^3 upper bound	Inequality	33	0.023109675 s/rad	Yes
τ^1 upper bound	Inequality	33	1.71E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	33	2.28E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	33	6.08E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	34	7.63E-07 rad^{-1}	No
q^2 upper bound	Inequality	34	9.18E-07 rad^{-1}	No
q^3 upper bound	Inequality	34	5.71E-06 rad^{-1}	No
ω^1 upper bound	Inequality	34	1.52E-06 s/rad	No
ω^2 upper bound	Inequality	34	0.018895776 s/rad	Yes
ω^3 upper bound	Inequality	34	0.022425417 s/rad	Yes
τ^1 upper bound	Inequality	34	1.69E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	34	2.26E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	34	6.16E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	35	7.74E-07 rad^{-1}	No
q^2 upper bound	Inequality	35	9.61E-07 rad^{-1}	No
q^3 upper bound	Inequality	35	7.93E-06 rad^{-1}	No
ω^1 upper bound	Inequality	35	1.84E-06 s/rad	No
ω^2 upper bound	Inequality	35	0.019722938 s/rad	Yes
ω^3 upper bound	Inequality	35	0.021042623 s/rad	Yes
τ^1 upper bound	Inequality	35	1.66E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	35	2.23E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	35	6.32E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	36	7.89E-07 rad^{-1}	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
q^2 upper bound	Inequality	36	1.01E-06 rad^{-1}	No
q^3 upper bound	Inequality	36	1.30E-05 rad^{-1}	No
ω^1 upper bound	Inequality	36	2.35E-06 s/rad	No
ω^2 upper bound	Inequality	36	0.016754458 s/rad	Yes
ω^3 upper bound	Inequality	36	0.016160744 s/rad	Yes
τ^1 upper bound	Inequality	36	1.63E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	36	2.19E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	36	6.37E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	37	8.09E-07 rad^{-1}	No
q^2 upper bound	Inequality	37	1.06E-06 rad^{-1}	No
q^3 upper bound	Inequality	37	3.26E-05 rad^{-1}	No
ω^1 upper bound	Inequality	37	3.17E-06 s/rad	No
ω^2 upper bound	Inequality	37	0.001880065 s/rad	Yes
ω^3 upper bound	Inequality	37	8.77E-06 s/rad	No
τ^1 upper bound	Inequality	37	1.59E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	37	2.08E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	37	5.71E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	38	8.32E-07 rad^{-1}	No
q^2 upper bound	Inequality	38	1.12E-06 rad^{-1}	No
q^3 upper bound	Inequality	38	0.0001654 rad^{-1}	Yes
ω^1 upper bound	Inequality	38	3.87E-06 s/rad	No
ω^2 upper bound	Inequality	38	0.009188909 s/rad	Yes
ω^3 upper bound	Inequality	38	1.16E-06 s/rad	No
τ^1 upper bound	Inequality	38	1.54E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	38	2.00E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	38	5.89E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	39	8.63E-07 rad^{-1}	No
q^2 upper bound	Inequality	39	1.16E-06 rad^{-1}	No
q^3 upper bound	Inequality	39	0.657356267 rad^{-1}	Yes
ω^1 upper bound	Inequality	39	0.002030711 s/rad	Yes
ω^2 upper bound	Inequality	39	1.47E-06 s/rad	No
ω^3 upper bound	Inequality	39	1.17E-06 s/rad	No
τ^1 upper bound	Inequality	39	1.47E-07 $(N\text{-}m)^{-1}$	No
τ^2 upper bound	Inequality	39	1.95E-07 $(N\text{-}m)^{-1}$	No
τ^3 upper bound	Inequality	39	5.72E-07 $(N\text{-}m)^{-1}$	No
q^1 upper bound	Inequality	40	9.01E-07 rad^{-1}	No
q^2 upper bound	Inequality	40	1.17E-06 rad^{-1}	No
q^3 upper bound	Inequality	40	0.00011039 rad^{-1}	Yes
ω^1 upper bound	Inequality	40	0.028639849 s/rad	Yes
ω^2 upper bound	Inequality	40	9.81E-07 s/rad	No
ω^3 upper bound	Inequality	40	6.81E-07 s/rad	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
τ^{1} upper bound	Inequality	40	1.42E-07 $(N\text{-}m)^{-1}$	No
τ^{2} upper bound	Inequality	40	1.91E-07 $(N\text{-}m)^{-1}$	No
τ^{3} upper bound	Inequality	40	5.64E-07 $(N\text{-}m)^{-1}$	No
q^{1} upper bound	Inequality	41	9.42E-07 rad^{-1}	No
q^{2} upper bound	Inequality	41	1.15E-06 rad^{-1}	No
q^{3} upper bound	Inequality	41	2.18E-05 rad^{-1}	No
ω^{1} upper bound	Inequality	41	0.036981036 s/rad	Yes
ω^{2} upper bound	Inequality	41	6.87E-07 s/rad	No
ω^{3} upper bound	Inequality	41	5.09E-07 s/rad	No
τ^{1} upper bound	Inequality	41	1.36E-07 $(N\text{-}m)^{-1}$	No
τ^{2} upper bound	Inequality	41	1.94E-07 $(N\text{-}m)^{-1}$	No
τ^{3} upper bound	Inequality	41	6.15E-07 $(N\text{-}m)^{-1}$	No
q^{1} upper bound	Inequality	42	9.88E-07 rad^{-1}	No
q^{2} upper bound	Inequality	42	1.11E-06 rad^{-1}	No
q^{3} upper bound	Inequality	42	1.05E-05 rad^{-1}	No
ω^{1} upper bound	Inequality	42	0.094920424 s/rad	Yes
ω^{2} upper bound	Inequality	42	5.47E-07 s/rad	No
ω^{3} upper bound	Inequality	42	5.09E-07 s/rad	No
τ^{1} upper bound	Inequality	42	1.36E-07 $(N\text{-}m)^{-1}$	No
τ^{2} upper bound	Inequality	42	1.99E-07 $(N\text{-}m)^{-1}$	No
τ^{3} upper bound	Inequality	42	6.31E-07 $(N\text{-}m)^{-1}$	No
q^{1} upper bound	Inequality	43	1.04E-06 rad^{-1}	No
q^{2} upper bound	Inequality	43	1.06E-06 rad^{-1}	No
q^{3} upper bound	Inequality	43	6.94E-06 rad^{-1}	No
ω^{1} upper bound	Inequality	43	0.087980777 s/rad	Yes
ω^{2} upper bound	Inequality	43	5.09E-07 s/rad	No
ω^{3} upper bound	Inequality	43	5.09E-07 s/rad	No
τ^{1} upper bound	Inequality	43	1.34E-07 $(N\text{-}m)^{-1}$	No
τ^{2} upper bound	Inequality	43	2.01E-07 $(N\text{-}m)^{-1}$	No
τ^{3} upper bound	Inequality	43	6.36E-07 $(N\text{-}m)^{-1}$	No
q^{1} upper bound	Inequality	44	1.09E-06 rad^{-1}	No
q^{2} upper bound	Inequality	44	1.00E-06 rad^{-1}	No
q^{3} upper bound	Inequality	44	5.17E-06 rad^{-1}	No
ω^{1} upper bound	Inequality	44	0.051856671 s/rad	Yes
ω^{2} upper bound	Inequality	44	5.09E-07 s/rad	No
ω^{3} upper bound	Inequality	44	5.09E-07 s/rad	No
τ^{1} upper bound	Inequality	44	1.29E-07 $(N\text{-}m)^{-1}$	No
τ^{2} upper bound	Inequality	44	1.98E-07 $(N\text{-}m)^{-1}$	No
τ^{3} upper bound	Inequality	44	6.19E-07 $(N\text{-}m)^{-1}$	No
q^{1} upper bound	Inequality	45	1.16E-06 rad^{-1}	No
q^{2} upper bound	Inequality	45	9.58E-07 rad^{-1}	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
q^{3} upper bound	Inequality	45	4.13E-06 rad^{-1}	No
ω^{1} upper bound	Inequality	45	0.066453521 s/rad	Yes
ω^{2} upper bound	Inequality	45	5.09E-07 s/rad	No
ω^{3} upper bound	Inequality	45	5.09E-07 s/rad	No
τ^{1} upper bound	Inequality	45	1.25E-07 $(N\text{-}m)^{-1}$	No
τ^{2} upper bound	Inequality	45	1.95E-07 $(N\text{-}m)^{-1}$	No
τ^{3} upper bound	Inequality	45	6.02E-07 $(N\text{-}m)^{-1}$	No
q^{1} upper bound	Inequality	46	1.23E-06 rad^{-1}	No
q^{2} upper bound	Inequality	46	9.15E-07 rad^{-1}	No
q^{3} upper bound	Inequality	46	3.43E-06 rad^{-1}	No
ω^{1} upper bound	Inequality	46	0.085136642 s/rad	Yes
ω^{2} upper bound	Inequality	46	5.09E-07 s/rad	No
ω^{3} upper bound	Inequality	46	5.09E-07 s/rad	No
τ^{1} upper bound	Inequality	46	1.21E-07 $(N\text{-}m)^{-1}$	No
τ^{2} upper bound	Inequality	46	1.92E-07 $(N\text{-}m)^{-1}$	No
τ^{3} upper bound	Inequality	46	5.86E-07 $(N\text{-}m)^{-1}$	No
q^{1} upper bound	Inequality	47	1.31E-06 rad^{-1}	No
q^{2} upper bound	Inequality	47	8.75E-07 rad^{-1}	No
q^{3} upper bound	Inequality	47	2.93E-06 rad^{-1}	No
ω^{1} upper bound	Inequality	47	0.009437391 s/rad	Yes
ω^{2} upper bound	Inequality	47	5.09E-07 s/rad	No
ω^{3} upper bound	Inequality	47	5.09E-07 s/rad	No
τ^{1} upper bound	Inequality	47	1.14E-07 $(N\text{-}m)^{-1}$	No
τ^{2} upper bound	Inequality	47	1.85E-07 $(N\text{-}m)^{-1}$	No
τ^{3} upper bound	Inequality	47	5.64E-07 $(N\text{-}m)^{-1}$	No
q^{1} upper bound	Inequality	48	1.39E-06 rad^{-1}	No
q^{2} upper bound	Inequality	48	8.39E-07 rad^{-1}	No
q^{3} upper bound	Inequality	48	2.56E-06 rad^{-1}	No
ω^{1} upper bound	Inequality	48	1.82E-05 s/rad	No
ω^{2} upper bound	Inequality	48	5.09E-07 s/rad	No
ω^{3} upper bound	Inequality	48	5.09E-07 s/rad	No
τ^{1} upper bound	Inequality	48	1.07E-07 $(N\text{-}m)^{-1}$	No
τ^{2} upper bound	Inequality	48	1.78E-07 $(N\text{-}m)^{-1}$	No
τ^{3} upper bound	Inequality	48	5.39E-07 $(N\text{-}m)^{-1}$	No
q^{1} upper bound	Inequality	49	1.48E-06 rad^{-1}	No
q^{2} upper bound	Inequality	49	8.06E-07 rad^{-1}	No
q^{3} upper bound	Inequality	49	2.28E-06 rad^{-1}	No
ω^{1} upper bound	Inequality	49	5.90E-06 s/rad	No
ω^{2} upper bound	Inequality	49	5.09E-07 s/rad	No
ω^{3} upper bound	Inequality	49	5.09E-07 s/rad	No
τ^{1} upper bound	Inequality	49	9.69E-08 $(N\text{-}m)^{-1}$	No

Constraint	Type	Time Index	Lagrange Multiplier	Active
τ^{2} upper bound	Inequality	49	1.91E-07 $(N\text{-}m)^{-1}$	No
τ^{3} upper bound	Inequality	49	7.48E-07 $(N\text{-}m)^{-1}$	No
q^{1} upper bound	Inequality	50	1.53E-06 rad^{-1}	No
q^{2} upper bound	Inequality	50	7.90E-07 rad^{-1}	No
q^{3} upper bound	Inequality	50	2.16E-06 rad^{-1}	No
ω^{1} upper bound	Inequality	50	1.02E-06 s/rad	No
ω^{2} upper bound	Inequality	50	1.02E-06 s/rad	No
ω^{3} upper bound	Inequality	50	1.02E-06 s/rad	No
τ^{1} upper bound	Inequality	50	1.68E-07 $(N\text{-}m)^{-1}$	No
τ^{2} upper bound	Inequality	50	2.67E-07 $(N\text{-}m)^{-1}$	No
τ^{3} upper bound	Inequality	50	8.00E-07 $(N\text{-}m)^{-1}$	No
Δt upper bound	Inequality	-	0 s^{-1}	No
q^{1} collocation constraint	Equality	1	4.871088654 rad^{-1}	Yes
q^{2} collocation constraint	Equality	1	2.878710505 rad^{-1}	Yes
q^{3} collocation constraint	Equality	1	0.915116678 rad^{-1}	Yes
ω^{1} collocation constraint	Equality	1	1.482309732 s/rad	Yes
ω^{2} collocation constraint	Equality	1	0.898712577 s/rad	Yes
ω^{3} collocation constraint	Equality	1	0.00293448 s/rad	Yes
q^{1} collocation constraint	Equality	2	4.282399838 rad^{-1}	Yes
q^{2} collocation constraint	Equality	2	2.545659242 rad^{-1}	Yes
q^{3} collocation constraint	Equality	2	1.141228294 rad^{-1}	Yes
ω^{1} collocation constraint	Equality	2	1.262997073 s/rad	Yes
ω^{2} collocation constraint	Equality	2	0.682190893 s/rad	Yes
ω^{3} collocation constraint	Equality	2	0.218122814 s/rad	Yes
q^{1} collocation constraint	Equality	3	3.747248823 rad^{-1}	Yes
q^{2} collocation constraint	Equality	3	2.265715853 rad^{-1}	Yes
q^{3} collocation constraint	Equality	3	1.178653402 rad^{-1}	Yes
ω^{1} collocation constraint	Equality	3	1.087567575 s/rad	Yes
ω^{2} collocation constraint	Equality	3	0.590447149 s/rad	Yes
ω^{3} collocation constraint	Equality	3	0.192165745 s/rad	Yes
q^{1} collocation constraint	Equality	4	3.265843756 rad^{-1}	Yes
q^{2} collocation constraint	Equality	4	1.986825296 rad^{-1}	Yes
q^{3} collocation constraint	Equality	4	1.12919943 rad^{-1}	Yes
ω^{1} collocation constraint	Equality	4	0.938260982 s/rad	Yes
ω^{2} collocation constraint	Equality	4	0.509889391 s/rad	Yes
ω^{3} collocation constraint	Equality	4	0.171542843 s/rad	Yes
q^{1} collocation constraint	Equality	5	2.834826912 rad^{-1}	Yes
q^{2} collocation constraint	Equality	5	1.716770032 rad^{-1}	Yes
q^{3} collocation constraint	Equality	5	1.041385264 rad^{-1}	Yes
ω^{1} collocation constraint	Equality	5	0.808181108 s/rad	Yes
ω^{2} collocation constraint	Equality	5	0.438523276 s/rad	Yes

Constraint	Type	Time Index	Lagrange Multiplier	Active
ω^3 collocation constraint	Equality	5	0.152571558 s/rad	Yes
q^1 collocation constraint	Equality	6	2.45080053 rad^{-1}	Yes
q^2 collocation constraint	Equality	6	1.467451242 rad^{-1}	Yes
q^3 collocation constraint	Equality	6	0.941408649 rad^{-1}	Yes
ω^1 collocation constraint	Equality	6	0.694068542 s/rad	Yes
ω^2 collocation constraint	Equality	6	0.375469571 s/rad	Yes
ω^3 collocation constraint	Equality	6	0.134554152 s/rad	Yes
q^1 collocation constraint	Equality	7	2.110730092 rad^{-1}	Yes
q^2 collocation constraint	Equality	7	1.244985454 rad^{-1}	Yes
q^3 collocation constraint	Equality	7	0.843375971 rad^{-1}	Yes
ω^1 collocation constraint	Equality	7	0.593903131 s/rad	Yes
ω^2 collocation constraint	Equality	7	0.320178992 s/rad	Yes
ω^3 collocation constraint	Equality	7	0.11759822 s/rad	Yes
q^1 collocation constraint	Equality	8	1.811427154 rad^{-1}	Yes
q^2 collocation constraint	Equality	8	1.050945123 rad^{-1}	Yes
q^3 collocation constraint	Equality	8	0.753914575 rad^{-1}	Yes
ω^1 collocation constraint	Equality	8	0.506291749 s/rad	Yes
ω^2 collocation constraint	Equality	8	0.272031746 s/rad	Yes
ω^3 collocation constraint	Equality	8	0.101887744 s/rad	Yes
q^1 collocation constraint	Equality	9	1.549514233 rad^{-1}	Yes
q^2 collocation constraint	Equality	9	0.88437045 rad^{-1}	Yes
q^3 collocation constraint	Equality	9	0.675588041 rad^{-1}	Yes
ω^1 collocation constraint	Equality	9	0.430068773 s/rad	Yes
ω^2 collocation constraint	Equality	9	0.230360648 s/rad	Yes
ω^3 collocation constraint	Equality	9	0.087576262 s/rad	Yes
q^1 collocation constraint	Equality	10	1.3217893 rad^{-1}	Yes
q^2 collocation constraint	Equality	10	0.742666064 rad^{-1}	Yes
q^3 collocation constraint	Equality	10	0.608907496 rad^{-1}	Yes
ω^1 collocation constraint	Equality	10	0.364086208 s/rad	Yes
ω^2 collocation constraint	Equality	10	0.19454752 s/rad	Yes
ω^3 collocation constraint	Equality	10	0.074801306 s/rad	Yes
q^1 collocation constraint	Equality	11	1.124809215 rad^{-1}	Yes
q^2 collocation constraint	Equality	11	0.623154701 rad^{-1}	Yes
q^3 collocation constraint	Equality	11	0.553041211 rad^{-1}	Yes
ω^1 collocation constraint	Equality	11	0.307318002 s/rad	Yes
ω^2 collocation constraint	Equality	11	0.163851262 s/rad	Yes
ω^3 collocation constraint	Equality	11	0.06350783 s/rad	Yes
q^1 collocation constraint	Equality	12	0.955384672 rad^{-1}	Yes
q^2 collocation constraint	Equality	12	0.522826405 rad^{-1}	Yes
q^3 collocation constraint	Equality	12	0.506705382 rad^{-1}	Yes
ω^1 collocation constraint	Equality	12	0.258669027 s/rad	Yes

Constraint	Type	Time Index	Lagrange Multiplier	Active
ω^{2} collocation constraint	Equality	12	0.137663872 s/rad	Yes
ω^{3} collocation constraint	Equality	12	0.053666324 s/rad	Yes
q^{1} collocation constraint	Equality	13	0.810390057 rad^{-1}	Yes
q^{2} collocation constraint	Equality	13	0.438941468 rad^{-1}	Yes
q^{3} collocation constraint	Equality	13	0.468501923 rad^{-1}	Yes
ω^{1} collocation constraint	Equality	13	0.217129472 s/rad	Yes
ω^{2} collocation constraint	Equality	13	0.115355815 s/rad	Yes
ω^{3} collocation constraint	Equality	13	0.045144978 s/rad	Yes
q^{1} collocation constraint	Equality	14	0.686840389 rad^{-1}	Yes
q^{2} collocation constraint	Equality	14	0.369104752 rad^{-1}	Yes
q^{3} collocation constraint	Equality	14	0.437212238 rad^{-1}	Yes
ω^{1} collocation constraint	Equality	14	0.181750146 s/rad	Yes
ω^{2} collocation constraint	Equality	14	0.096355748 s/rad	Yes
ω^{3} collocation constraint	Equality	14	0.037791681 s/rad	Yes
q^{1} collocation constraint	Equality	15	0.582080572 rad^{-1}	Yes
q^{2} collocation constraint	Equality	15	0.311124059 rad^{-1}	Yes
q^{3} collocation constraint	Equality	15	0.411775917 rad^{-1}	Yes
ω^{1} collocation constraint	Equality	15	0.151638608 s/rad	Yes
ω^{2} collocation constraint	Equality	15	0.080197055 s/rad	Yes
ω^{3} collocation constraint	Equality	15	0.031478874 s/rad	Yes
q^{1} collocation constraint	Equality	16	0.493726239 rad^{-1}	Yes
q^{2} collocation constraint	Equality	16	0.263138568 rad^{-1}	Yes
q^{3} collocation constraint	Equality	16	0.391336737 rad^{-1}	Yes
ω^{1} collocation constraint	Equality	16	0.126007833 s/rad	Yes
ω^{2} collocation constraint	Equality	16	0.066446273 s/rad	Yes
ω^{3} collocation constraint	Equality	16	0.026060583 s/rad	Yes
q^{1} collocation constraint	Equality	17	0.419661714 rad^{-1}	Yes
q^{2} collocation constraint	Equality	17	0.223618738 rad^{-1}	Yes
q^{3} collocation constraint	Equality	17	0.375241739 rad^{-1}	Yes
ω^{1} collocation constraint	Equality	17	0.104160146 s/rad	Yes
ω^{2} collocation constraint	Equality	17	0.054716487 s/rad	Yes
ω^{3} collocation constraint	Equality	17	0.021394354 s/rad	Yes
q^{1} collocation constraint	Equality	18	0.358057867 rad^{-1}	Yes
q^{2} collocation constraint	Equality	18	0.191308784 rad^{-1}	Yes
q^{3} collocation constraint	Equality	18	0.36298877 rad^{-1}	Yes
ω^{1} collocation constraint	Equality	18	0.085476126 s/rad	Yes
ω^{2} collocation constraint	Equality	18	0.044673123 s/rad	Yes
ω^{3} collocation constraint	Equality	18	0.017352494 s/rad	Yes
q^{1} collocation constraint	Equality	19	0.307385224 rad^{-1}	Yes
q^{2} collocation constraint	Equality	19	0.16514828 rad^{-1}	Yes
q^{3} collocation constraint	Equality	19	0.354146782 rad^{-1}	Yes

Constraint	Type	Time Index	Lagrange Multiplier	Active
ω^1 collocation constraint	Equality	19	0.069407471 s/rad	Yes
ω^2 collocation constraint	Equality	19	0.036031668 s/rad	Yes
ω^3 collocation constraint	Equality	19	0.013826403 s/rad	Yes
q^1 collocation constraint	Equality	20	0.266147196 rad^{-1}	Yes
q^2 collocation constraint	Equality	20	0.144675395 rad^{-1}	Yes
q^3 collocation constraint	Equality	20	0.348647465 rad^{-1}	Yes
ω^1 collocation constraint	Equality	20	0.055508254 s/rad	Yes
ω^2 collocation constraint	Equality	20	0.028441912 s/rad	Yes
ω^3 collocation constraint	Equality	20	0.010645434 s/rad	Yes
q^1 collocation constraint	Equality	21	0.233187873 rad^{-1}	Yes
q^2 collocation constraint	Equality	21	0.129488837 rad^{-1}	Yes
q^3 collocation constraint	Equality	21	0.346332089 rad^{-1}	Yes
ω^1 collocation constraint	Equality	21	0.04332213 s/rad	Yes
ω^2 collocation constraint	Equality	21	0.021681749 s/rad	Yes
ω^3 collocation constraint	Equality	21	0.007728408 s/rad	Yes
q^1 collocation constraint	Equality	22	0.207269222 rad^{-1}	Yes
q^2 collocation constraint	Equality	22	0.119968226 rad^{-1}	Yes
q^3 collocation constraint	Equality	22	0.347497384 rad^{-1}	Yes
ω^1 collocation constraint	Equality	22	0.032516859 s/rad	Yes
ω^2 collocation constraint	Equality	22	0.015364276 s/rad	Yes
ω^3 collocation constraint	Equality	22	0.004851904 s/rad	Yes
q^1 collocation constraint	Equality	23	0.186521203 rad^{-1}	Yes
q^2 collocation constraint	Equality	23	0.118661535 rad^{-1}	Yes
q^3 collocation constraint	Equality	23	0.353790289 rad^{-1}	Yes
ω^1 collocation constraint	Equality	23	0.022939805 s/rad	Yes
ω^2 collocation constraint	Equality	23	0.00875347 s/rad	Yes
ω^3 collocation constraint	Equality	23	0.001462403 s/rad	Yes
q^1 collocation constraint	Equality	24	0.169742499 rad^{-1}	Yes
q^2 collocation constraint	Equality	24	0.12804682 rad^{-1}	Yes
q^3 collocation constraint	Equality	24	0.366233292 rad^{-1}	Yes
ω^1 collocation constraint	Equality	24	0.01442364 s/rad	Yes
ω^2 collocation constraint	Equality	24	0.001685945 s/rad	Yes
ω^3 collocation constraint	Equality	24	-0.002568561 s/rad	Yes
q^1 collocation constraint	Equality	25	0.157543833 rad^{-1}	Yes
q^2 collocation constraint	Equality	25	0.145374724 rad^{-1}	Yes
q^3 collocation constraint	Equality	25	0.382433206 rad^{-1}	Yes
ω^1 collocation constraint	Equality	25	0.006698513 s/rad	Yes
ω^2 collocation constraint	Equality	25	-0.00485095 s/rad	Yes
ω^3 collocation constraint	Equality	25	-0.006326339 s/rad	Yes
q^1 collocation constraint	Equality	26	0.150087219 rad^{-1}	Yes
q^2 collocation constraint	Equality	26	0.169534314 rad^{-1}	Yes

Constraint	Type	Time Index	Lagrange Multiplier	Active
q^3 collocation constraint	Equality	26	0.401336037 rad^{-1}	Yes
ω^1 collocation constraint	Equality	26	-0.000537908 s/rad	Yes
ω^2 collocation constraint	Equality	26	-0.01149348 s/rad	Yes
ω^3 collocation constraint	Equality	26	-0.010231338 s/rad	Yes
q^1 collocation constraint	Equality	27	0.14762283 rad^{-1}	Yes
q^2 collocation constraint	Equality	27	0.19942501 rad^{-1}	Yes
q^3 collocation constraint	Equality	27	0.421748175 rad^{-1}	Yes
ω^1 collocation constraint	Equality	27	-0.007593345 s/rad	Yes
ω^2 collocation constraint	Equality	27	-0.018414122 s/rad	Yes
ω^3 collocation constraint	Equality	27	-0.01428526 s/rad	Yes
q^1 collocation constraint	Equality	28	0.150524747 rad^{-1}	Yes
q^2 collocation constraint	Equality	28	0.233937142 rad^{-1}	Yes
q^3 collocation constraint	Equality	28	0.442307246 rad^{-1}	Yes
ω^1 collocation constraint	Equality	28	-0.014788113 s/rad	Yes
ω^2 collocation constraint	Equality	28	-0.025723523 s/rad	Yes
ω^3 collocation constraint	Equality	28	-0.018426243 s/rad	Yes
q^1 collocation constraint	Equality	29	0.159361058 rad^{-1}	Yes
q^2 collocation constraint	Equality	29	0.271972903 rad^{-1}	Yes
q^3 collocation constraint	Equality	29	0.46149024 rad^{-1}	Yes
ω^1 collocation constraint	Equality	29	-0.02247706 s/rad	Yes
ω^2 collocation constraint	Equality	29	-0.033489801 s/rad	Yes
ω^3 collocation constraint	Equality	29	-0.022541864 s/rad	Yes
q^1 collocation constraint	Equality	30	0.174967005 rad^{-1}	Yes
q^2 collocation constraint	Equality	30	0.312494376 rad^{-1}	Yes
q^3 collocation constraint	Equality	30	0.477594716 rad^{-1}	Yes
ω^1 collocation constraint	Equality	30	-0.031031232 s/rad	Yes
ω^2 collocation constraint	Equality	30	-0.041729248 s/rad	Yes
ω^3 collocation constraint	Equality	30	-0.026460456 s/rad	Yes
q^1 collocation constraint	Equality	31	0.198593044 rad^{-1}	Yes
q^2 collocation constraint	Equality	31	0.354515164 rad^{-1}	Yes
q^3 collocation constraint	Equality	31	0.488724456 rad^{-1}	Yes
ω^1 collocation constraint	Equality	31	-0.040865765 s/rad	Yes
ω^2 collocation constraint	Equality	31	-0.050403625 s/rad	Yes
ω^3 collocation constraint	Equality	31	-0.029948011 s/rad	Yes
q^1 collocation constraint	Equality	32	0.232098091 rad^{-1}	Yes
q^2 collocation constraint	Equality	32	0.397085059 rad^{-1}	Yes
q^3 collocation constraint	Equality	32	0.492733131 rad^{-1}	Yes
ω^1 collocation constraint	Equality	32	-0.052443703 s/rad	Yes
ω^2 collocation constraint	Equality	32	-0.05941224 s/rad	Yes
ω^3 collocation constraint	Equality	32	-0.032697613 s/rad	Yes
q^1 collocation constraint	Equality	33	0.278227737 rad^{-1}	Yes

Constraint	Type	Time Index	Lagrange Multiplier	Active
q^2 collocation constraint	Equality	33	0.439256295 rad^{-1}	Yes
q^3 collocation constraint	Equality	33	0.487152224 rad^{-1}	Yes
ω^1 collocation constraint	Equality	33	-0.066319356 s/rad	Yes
ω^2 collocation constraint	Equality	33	-0.068628552 s/rad	Yes
ω^3 collocation constraint	Equality	33	-0.034355238 s/rad	Yes
q^1 collocation constraint	Equality	34	0.340999339 rad^{-1}	Yes
q^2 collocation constraint	Equality	34	0.479896624 rad^{-1}	Yes
q^3 collocation constraint	Equality	34	0.469020613 rad^{-1}	Yes
ω^1 collocation constraint	Equality	34	-0.083173954 s/rad	Yes
ω^2 collocation constraint	Equality	34	-0.077888212 s/rad	Yes
ω^3 collocation constraint	Equality	34	-0.034502154 s/rad	Yes
q^1 collocation constraint	Equality	35	0.42621605 rad^{-1}	Yes
q^2 collocation constraint	Equality	35	0.51723702 rad^{-1}	Yes
q^3 collocation constraint	Equality	35	0.434569758 rad^{-1}	Yes
ω^1 collocation constraint	Equality	35	-0.10389142 s/rad	Yes
ω^2 collocation constraint	Equality	35	-0.087026616 s/rad	Yes
ω^3 collocation constraint	Equality	35	-0.032668964 s/rad	Yes
q^1 collocation constraint	Equality	36	0.539511556 rad^{-1}	Yes
q^2 collocation constraint	Equality	36	0.546145427 rad^{-1}	Yes
q^3 collocation constraint	Equality	36	0.368985055 rad^{-1}	Yes
ω^1 collocation constraint	Equality	36	-0.129914038 s/rad	Yes
ω^2 collocation constraint	Equality	36	-0.099413582 s/rad	Yes
ω^3 collocation constraint	Equality	36	-0.030929291 s/rad	Yes
q^1 collocation constraint	Equality	37	0.678456328 rad^{-1}	Yes
q^2 collocation constraint	Equality	37	0.538567141 rad^{-1}	Yes
q^3 collocation constraint	Equality	37	0.191989085 rad^{-1}	Yes
ω^1 collocation constraint	Equality	37	-0.164524317 s/rad	Yes
ω^2 collocation constraint	Equality	37	-0.127970741 s/rad	Yes
ω^3 collocation constraint	Equality	37	-0.037804791 s/rad	Yes
q^1 collocation constraint	Equality	38	0.865239634 rad^{-1}	Yes
q^2 collocation constraint	Equality	38	0.534244795 rad^{-1}	Yes
q^3 collocation constraint	Equality	38	-0.055639941 rad^{-1}	Yes
ω^1 collocation constraint	Equality	38	-0.211769621 s/rad	Yes
ω^2 collocation constraint	Equality	38	-0.150915758 s/rad	Yes
ω^3 collocation constraint	Equality	38	-0.034568708 s/rad	Yes
q^1 collocation constraint	Equality	39	1.112612723 rad^{-1}	Yes
q^2 collocation constraint	Equality	39	0.41770239 rad^{-1}	Yes
q^3 collocation constraint	Equality	39	0.221768742 rad^{-1}	Yes
ω^1 collocation constraint	Equality	39	-0.274573814 s/rad	Yes
ω^2 collocation constraint	Equality	39	-0.179565662 s/rad	Yes
ω^3 collocation constraint	Equality	39	-0.032945719 s/rad	Yes

Constraint	Type	Time Index	Lagrange Multiplier	Active
q^1 collocation constraint	Equality	40	1.4052451 rad^{-1}	Yes
q^2 collocation constraint	Equality	40	0.242251757 rad^{-1}	Yes
q^3 collocation constraint	Equality	40	-0.229835554 rad^{-1}	Yes
ω^1 collocation constraint	Equality	40	-0.333287917 s/rad	Yes
ω^2 collocation constraint	Equality	40	-0.206517321 s/rad	Yes
ω^3 collocation constraint	Equality	40	-0.03322721 s/rad	Yes
q^1 collocation constraint	Equality	41	1.770033691 rad^{-1}	Yes
q^2 collocation constraint	Equality	41	0.048482103 rad^{-1}	Yes
q^3 collocation constraint	Equality	41	-0.60469547 rad^{-1}	Yes
ω^1 collocation constraint	Equality	41	-0.410392055 s/rad	Yes
ω^2 collocation constraint	Equality	41	-0.229830107 s/rad	Yes
ω^3 collocation constraint	Equality	41	-0.028644969 s/rad	Yes
q^1 collocation constraint	Equality	42	2.14878986 rad^{-1}	Yes
q^2 collocation constraint	Equality	42	-0.11506777 rad^{-1}	Yes
q^3 collocation constraint	Equality	42	-0.898827252 rad^{-1}	Yes
ω^1 collocation constraint	Equality	42	-0.452316416 s/rad	Yes
ω^2 collocation constraint	Equality	42	-0.246697075 s/rad	Yes
ω^3 collocation constraint	Equality	42	-0.033888476 s/rad	Yes
q^1 collocation constraint	Equality	43	2.563428047 rad^{-1}	Yes
q^2 collocation constraint	Equality	43	-0.260056245 rad^{-1}	Yes
q^3 collocation constraint	Equality	43	-1.154847312 rad^{-1}	Yes
ω^1 collocation constraint	Equality	43	-0.525203743 s/rad	Yes
ω^2 collocation constraint	Equality	43	-0.279379892 s/rad	Yes
ω^3 collocation constraint	Equality	43	-0.043116984 s/rad	Yes
q^1 collocation constraint	Equality	44	3.053454978 rad^{-1}	Yes
q^2 collocation constraint	Equality	44	-0.425505133 rad^{-1}	Yes
q^3 collocation constraint	Equality	44	-1.448438995 rad^{-1}	Yes
ω^1 collocation constraint	Equality	44	-0.664037626 s/rad	Yes
ω^2 collocation constraint	Equality	44	-0.348456498 s/rad	Yes
ω^3 collocation constraint	Equality	44	-0.061693898 s/rad	Yes
q^1 collocation constraint	Equality	45	3.618752258 rad^{-1}	Yes
q^2 collocation constraint	Equality	45	-0.608764128 rad^{-1}	Yes
q^3 collocation constraint	Equality	45	-1.777956513 rad^{-1}	Yes
ω^1 collocation constraint	Equality	45	-0.821811585 s/rad	Yes
ω^2 collocation constraint	Equality	45	-0.427569626 s/rad	Yes
ω^3 collocation constraint	Equality	45	-0.085758531 s/rad	Yes
q^1 collocation constraint	Equality	46	4.257645471 rad^{-1}	Yes
q^2 collocation constraint	Equality	46	-0.805566994 rad^{-1}	Yes
q^3 collocation constraint	Equality	46	-2.140072301 rad^{-1}	Yes
ω^1 collocation constraint	Equality	46	-0.997852068 s/rad	Yes
ω^2 collocation constraint	Equality	46	-0.516351844 s/rad	Yes

Constraint	Type	Time Index	Lagrange Multiplier	Active
ω^3 collocation constraint	Equality	46	-0.115565806 s/rad	Yes
q^1 collocation constraint	Equality	47	5.035322135 rad^{-1}	Yes
q^2 collocation constraint	Equality	47	-1.085423401 rad^{-1}	Yes
q^3 collocation constraint	Equality	47	-2.616760381 rad^{-1}	Yes
ω^1 collocation constraint	Equality	47	-1.297520889 s/rad	Yes
ω^2 collocation constraint	Equality	47	-0.666976887 s/rad	Yes
ω^3 collocation constraint	Equality	47	-0.162987364 s/rad	Yes
q^1 collocation constraint	Equality	48	5.967550562 rad^{-1}	Yes
q^2 collocation constraint	Equality	48	-1.463923967 rad^{-1}	Yes
q^3 collocation constraint	Equality	48	-3.235977552 rad^{-1}	Yes
ω^1 collocation constraint	Equality	48	-1.661817898 s/rad	Yes
ω^2 collocation constraint	Equality	48	-0.851619737 s/rad	Yes
ω^3 collocation constraint	Equality	48	-0.225477695 s/rad	Yes
q^1 collocation constraint	Equality	49	7.177641998 rad^{-1}	Yes
q^2 collocation constraint	Equality	49	-1.911442017 rad^{-1}	Yes
q^3 collocation constraint	Equality	49	-3.319464621 rad^{-1}	Yes
ω^1 collocation constraint	Equality	49	-2.092011331 s/rad	Yes
ω^2 collocation constraint	Equality	49	-1.029257996 s/rad	Yes
ω^3 collocation constraint	Equality	49	-0.270606085 s/rad	Yes

Appendix D - Code

This is the MATLAB code used throughout this project. The file *Dynamics_2D.m* is used to generate the matrix functions for the Lagrangian dynamic equations of motion. The file *ArmOptimization.m* is the main file used for testing and solving our optimization problem. Two additional function files are included. One, *gensplines.m*, generates random polynomial splines used as initial guesses for the optimization. The other, *animate_2D.m*, visualizes the solution on a simplified 2D Kuka youBot.

```
%{
ArmOptimization.m
3-DOF arm optimization with random feasible starting trajectory
%}

clear all;
close all force;
clc;

%% Optimiza arm trajectory
% Arm initial conditions
q0 = [-pi/6 -pi/6 -pi/3].';
w0 = [0 0 0].';

% Arm final conditions
qf = [pi/6 pi/6 pi/3].';
wf = [0 0 0].';

% Arm joint limits
qmin = [-65 -151 -102.5].' * pi/180;
qmax = [90 146 102.5].' * pi/180;
wmin = [-90 -90 -90].' * pi/180;
wmax = [90 90 90].' * pi/180;
taumin = [-9.5 -6 -2].';
taumax = [9.5 6 2].';
lim_tol = 1e-4;

% Arm simulation time
dt = 0.1;
t0 = 0;
N = 50;
tspan = t0 : dt : ((N-1)*dt+t0);

% Equality constraints for initial & final conditions
Acond = [eye(6) zeros(6,9*N-6); zeros(6,9*N-9) eye(6) zeros(6,3)];
Acond = [Acond, zeros(12, 1)];
bcond = [q0; w0; qf; wf];

% Solver options
opt = optimoptions('fmincon');
opt.Display = 'off';  % Set to 'off' if running a bunch
opt.Algorithm = 'interior-point';
opt.MaxFunctionEvaluations = Inf;%1e5
opt.MaxIterations = 300;        % Play around with this number
opt.UseParallel = true;
% opt.OutputFcn = @fmincon_of;    % Plots the function values during the optimization (comment out if running ↙
more than once)

% Solver setup
problem.lb = [repmat([qmin; wmin; taumin], N, 1); 0];
problem.ub = [repmat([qmax; wmax; taumax], N, 1); Inf];
problem.Aeq = Acond;
problem.beq = bcond;
problem.nonlcon = @(x) constr(x, N);
problem.solver = 'fmincon';
problem.options = opt;

% Storage setup
info.x0 = [];       % initial conditions
info.p0 = [];       % polynomial orders
info.flag0 = [];    % polynomial exit flags
info.xsol = [];     % trajectory solutions (blocked)
info.dtsol = [];    % time solutions
info.fsol = [];     % objective values
info.ftimesol = []; % time costs
```

```
info.fworksol = []; % work costs
info.beta = [];     % weight
info.flagsol = [];  % optimization exit flags

beta = 0.5;

ind = 1;
while(ind <= 1)
    disp(ind);

    % Generate feasible random starting trajectories
    ord_seed = floor(3 * rand(1, 1)) + 4;
    [q10, w10, a10, flag1] = gensplines(1, ord_seed, tspan, qmin(1)+lim_tol, qmax(1)-lim_tol, wmin(1)+lim_tol, ↙
wmax(1)-lim_tol, q0(1), qf(1), w0(1), wf(1));
    [q20, w20, a20, flag2] = gensplines(1, ord_seed, tspan, qmin(2)+lim_tol, qmax(2)-lim_tol, wmin(2)+lim_tol, ↙
wmax(2)-lim_tol, q0(2), qf(2), w0(2), wf(2));
    [q30, w30, a30, flag3] = gensplines(1, ord_seed, tspan, qmin(3)+lim_tol, qmax(3)-lim_tol, wmin(3)+lim_tol, ↙
wmax(3)-lim_tol, q0(3), qf(3), w0(3), wf(3));

    % Get corresponding torque trajectories
    tau10 = zeros(1, N);
    tau20 = zeros(1, N);
    tau30 = zeros(1, N);
    for i = 1 : N
        q0i = [q10(i) q20(i) q30(i)].';
        w0i = [w10(i) w20(i) w30(i)].';
        a0i = [a10(i) a20(i) a30(i)].';

        tau0vec = MR_2Dfunc(q0i)*a0i + CR_2Dfunc(q0i, w0i)*w0i + NR_2Dfunc(q0i);
        tau10(i) = tau0vec(1);
        tau20(i) = tau0vec(2);
        tau30(i) = tau0vec(3);
    end

    % Optimization initial conditions
    x0 = [q10; q20; q30; w10; w20; w30; tau10; tau20; tau30];    % Square version
    x0vec = [reshape(x0, 9*N, 1); dt];                           % Vectorized version

    problem.x0 = x0vec;
    problem.objective = @(x)obj(x, N, beta);

    [xsolvec, fval, flag, outinfo, lambda, ~, ~] = fmincon(problem);
    xsol = reshape(xsolvec(1:end-1), 9, N);
    dtsol = xsolvec(end);

    [T, W] = multiobj([reshape(xsol, 9*N, 1); dtsol], N);

    info.x0(:, :, ind) = x0;
    info.p0(:, ind) = ord_seed;
    info.flag0(:, ind) = [flag1; flag2; flag3];
    info.xsol(:, :, ind) = xsol;
    info.dtsol(ind) = dtsol;
    info.fsol(ind) = fval;
    info.ftimesol(ind) = T;
    info.fworksol(ind) = W;
    info.beta(ind) = beta;
    info.flagsol(ind) = flag;

%     save('optinfo.mat', 'info');    % save each time in case something happens

    ind = ind + 1;
end

%% Helper functions for optimization
function f = obj(xvec, N, beta)
```

```
    [T, W] = multiobj(xvec, N);
    f = beta*T + (1-beta)*W;
end

function[T, W] = multiobj(xvec, N)
    x = reshape(xvec(1:end-1), 9, N);
    dt = xvec(end);

    W = 0;
    for i = 1 : size(x, 2)-1
        tau_i = x(7:9, i);
        tau_i1 = x(7:9, i+1);

        wi = dt * (tau_i.'*tau_i + tau_i1.'*tau_i1) / 2;
        W = W+ wi;
    end

    T = N * dt;
end

function [c, ceq] = constr(xvec, N)
    x = reshape(xvec(1:end-1), 9, N);
    dt = xvec(end);

    c = [];
    ceq = [];

    M = MR_2Dfunc(x(1:3, 1));
    C = CR_2Dfunc(x(1:3, 1), x(4:6, 1));
    N = NR_2Dfunc(x(1:3, 1));
    tau = x(7:9, 1);
    U = UR_2Dfunc(tau);

    b = 0.1 * eye(3);   % Arbitrary joint damping
    ddq_i = M \ (U - C*x(4:6,1) - N - b*x(4:6,1));

    for i = 1 : length(x(1,:))-1

        x_i = [x(1:3,i); x(4:6,i)];
        x_i1 = [x(1:3,i+1); x(4:6,i+1)];
        dx_i = [x(4:6,i); ddq_i];

        M = MR_2Dfunc(x(1:3,i));
        C = CR_2Dfunc(x(1:3,i), x(4:6,i));
        N = NR_2Dfunc(x(1:3,i));
        tau = x(7:9,i);
        U = UR_2Dfunc(tau);
        ddq_i1 = M \ (U - C*x(4:6,i) - N - b*x(4:6,i));

        dx_i1 = [x(4:6,i+1); ddq_i1];

        xout = x_i + dt * 0.5 * (dx_i+dx_i1);
        ceq = [ceq;x_i1-xout];

    end
end

function[stop] = fmincon_of(~, info, state)
    global fvals;

    switch state
        case 'init'
            fvals = [fvals info.fval];

            figure();
```

```
            hold on;
            xlabel('Iteration');
            ylabel('Objective');

        case 'iter'
            if(size(info.fval, 1) ~= 0)
                fvals = [fvals info.fval];

                plot(info.iteration+[0 1], fvals(end-1:end), '-k');
                drawnow;
            end
    end

    stop = false;
end
```

```
%{
gensplines.m
"Automatic" trajectory spline generation

Creates a pth-order polynomial that satisfies the initial and final joint
positions and velocities, as well as keeps them within the joint limits.
There are (p-3) free parameters with this method, so the spline can be
4th-order or higher (technically 3rd, but then there's only 1 feasible
spline since there's no free parameters).

This method is something that can actually be used. It essentially solves
an optimization without a cost function to select the polynomial
coefficients. So it stops once it finds any feasible point.

Inputs:
n       - Number of splines to generate
p       - Polynomial order
tvec    - Row vector of discrete time points
qmin    - Joint position lower bound
qmax    - Joint position upper bound
dqmin   - Joint velocity lower bound
dqmax   - Joint velocity upper bound
q0      - Initial joint position
qf      - Final joint position
dq0     - Initial joint velocity
dqf     - Final joint velocity

Outputs:
qvec    - Matrix of position trajectories where each row is a trajectory
dqvec   - Matrix of velocity trajectories where each row is a trajectory
ddqvec   - Matrix of acceleration trajectories where each row is a trajectory
flagvec - Vector of fmincon exit flags for each spline generation
%}

function[qvecs, dqvecs, ddqvecs, flagvec] = gensplines(ngen, p, tvec, qmin, qmax, dqmin, dqmax, q0, qf, dq0, ↙
dqf)
    % Polynomial order too small
    if(p == 3)
        warning('No free polynomial coefficients. A maximum of 1 feasible trajectory is possible. Set p to 4 or ↙
greater.');
    elseif(p < 3)
        error('Polynomial order is too low to satisfy the constraints. Set p to 4 or greater.' );
    end

    % Force time vector to be a row
    if(iscolumn(tvec))
        tvec = tvec.';
    end

    % Force time vector to start at 0
    if(tvec(1) ~= 0)
        tvec = tvec - tvec(1);
    end

    % Extract the last time value
    tf = tvec(end);

    % Initial condition constraints
    aeq1 = [1 zeros(1,p)];  % initial position constraint
    beq1 = q0;
    aeq2 = [0 1 zeros(1,p-1)];  % initial velocity constraint
    beq2 = dq0;

    % Final condition constraints
    aeq3 = [1 zeros(1,p)];
```

```
aeq4 = zeros(1, p+1);
for k = 1 : p
    aeq3(k+1) = tf^k;
    aeq4(k+1) = k * tf^(k-1);
end
beq3 = qf;
beq4 = dqf;

% Block of time vector powers for joint position limit inequalities
tblock = zeros(length(tvec), p+1);
for k = 0 : p
    tblock(:, k+1) = tvec.' .^ k;
end

% Joint position limit constraints
aineq1 = -tblock;                            % lower limit
bineq1 = repmat(-qmin, length(tvec), 1);
aineq2 = tblock;                             % upper limit
bineq2 = repmat(qmax, length(tvec), 1);

% Modified block of time vector powers for joint velocity limit inequalities
tblock(:, 2:end) = tblock(:, 1:(end-1));
tblock(:, 1) = zeros(length(tvec), 1);

% Joint velocity limit constraints
aineq3 = -tblock * diag(0:p).';              % lower limit
bineq3 = repmat(-dqmin, length(tvec), 1);
aineq4 = tblock * diag(0:p).';               % upper limit
bineq4 = repmat(dqmax, length(tvec), 1);

% Concatenate all of the linear equality & inequality constraints
Aeq = [aeq1; aeq2; aeq3; aeq4];
Beq = [beq1; beq2; beq3; beq4];
Aineq = [aineq1; aineq2; aineq3; aineq4];
Bineq = [bineq1; bineq2; bineq3; bineq4];

% Optimization options
opt = optimoptions('fmincon', 'Display', 'off', ...
    'Algorithm', 'active-set', ...
    'SpecifyObjectiveGradient', true, ...
    'MaxFunctionEvaluations', (p+1)*1000);

% Initialize return containers
qvecs = zeros(ngen, length(tvec));
dqvecs = zeros(ngen, length(tvec));
ddqvecs = zeros(ngen, length(tvec));
flagvec = zeros(ngen, 1);

% Generate some random splines
for i = 1 : ngen
    % Solve for a feasible set of polynomial coefficients
    c0 = (rand(p+1, 1) - 0.5) * 10;
    [ci, ~, flagi, ~] = fmincon(@cost, c0, Aineq, Bineq, Aeq, Beq, [], [], [], opt);

    % Get the spline
    [qveci, dqveci, ddqveci] = gentraj(ci, tvec);

    % Store return data
    qvecs(i, :) = qveci;
    dqvecs(i, :) = dqveci;
    ddqvecs(i, :) = ddqveci;
    flagvec(i) = flagi;

    % Infeasible spline found
    if(flagi == -2)
```

```
            warning('Spline generation %d did not find a feasible point. Consider using a longer timespan.' , i);
        end
    end
end

% Optimization cost function & gradient (just zero since don't care)
function[f, gradf] = cost(c)
    f = 0;
    gradf = zeros(1, length(c));
end

% Helper function to generate the spline given the polynomial coefficients and time vector
function[qvec, dqvec, ddqvec, tauvec] = gentraj(c, tvec)
    % Extract the polynomial order
    p = length(c) - 1;

    % Initialize the trajectories with the 0th & 1st order term
    qvec = c(1) * ones(size(tvec)) + c(2) * tvec;
    dqvec = c(2) * ones(size(tvec));
    ddqvec = zeros(size(tvec));

    % Add on each k-th order term
    for k = 2 : p
        qvec = qvec + (c(k+1) * tvec.^k);
        dqvec = dqvec + (k * c(k+1) * tvec.^(k-1));
        ddqvec = ddqvec + (k*(k-1) * c(k+1) * tvec.^(k-2));
    end
end
```

```
%{
Dynamics_2D.m
2D dynamics generation for 3-link robotic arm, joints 2, 3, & 4
%}

clear all;
close all force;
clc;

%% Robotic arm dynamics derivation
% Link lengths (m)
d0 = 151 / 1000;
d1 = 33 / 1000;
h1 = 147 / 1000;
d2 = 155 / 1000;
d3 = 135 / 1000;
d4 = 217.5 / 1000;

% Link distances from joint to COM (m)
theta2 = 77.5 / 1000;
theta3 = 67.5 / 1000;
theta4 = 108.75 / 1000;

% Link linear inertias (kg)
m2L = 1.318;
m3L = 0.821;
m4L = 0.769;

% Link rotational intertias (kg-m^2)
m2XX = 0.0031145;
m3XX = 0.00172767;
m4XX = 0.0006764;

% Mass matrices
M2 = diag([m2L m2L m2XX]);
M3 = diag([m3L m3L m3XX]);
M4 = diag([m4L m4L m4XX]);

% Robot position states
syms q2 q3 q4 real;
q = [q2 q3 q4].';

% Robot velocity states
syms w2 w3 w4 real;
dq = [w2 w3 w4].';

% Transformation matrices
g_P_S = [0 -1 d0+d1;    % platform to joint 2
         1  0    h1;
         0  0     1];
g_S_L2 = [cos(q2) -sin(q2) theta2*cos(q2);   % joint 2 to COM 2
          sin(q2)  cos(q2) theta2*sin(q2);
                0        0              1];
g_L2_L3 = [cos(q3) -sin(q3) (d2-theta2)+theta3*cos(q3); % COM 2 to COM 3
           sin(q3)  cos(q3)             theta3*sin(q3);
                 0        0                          1];
g_L3_L4 = [cos(q4) -sin(q4) (d3-theta3)+theta4*cos(q4); % COM 3 to COM 4
           sin(q4)  cos(q4)             theta4*sin(q4);
                 0        0                          1];
g_L4_T = [1 0 (d4-theta4);  % COM 4 to gripper
          0 1           0;
          0 0           1];
g_S_L3 = simplify(g_S_L2 * g_L2_L3);     % joint 2 to COM 3
g_S_L4 = simplify(g_S_L3 * g_L3_L4);     % joint 2 to COM 4
g_S_T = simplify(g_S_L4 * g_L4_T);       % joint 2 to gripper
```

```
g_P_T = simplify(g_P_S * g_S_T);        % platform to gripper

% Body Jacobian matrices
Jb_S_L2 = simplify([rbvel2twist2D(g_S_L2\diff(g_S_L2, q2)), ...
                                                zeros(3, 1), ...
                                                zeros(3, 1)]);
Jb_S_L3 = simplify([rbvel2twist2D(g_S_L3\diff(g_S_L3, q2)), ...
                    rbvel2twist2D(g_S_L3\diff(g_S_L3, q3)), ...
                                                zeros(3, 1)]);
Jb_S_L4 = simplify([rbvel2twist2D(g_S_L4\diff(g_S_L4, q2)), ...
                    rbvel2twist2D(g_S_L4\diff(g_S_L4, q3)), ...
                    rbvel2twist2D(g_S_L4\diff(g_S_L4, q4))]);

% Lagrangian mass matrix
MR = simplify(Jb_S_L2.'*M2*Jb_S_L2 + Jb_S_L3.'*M3*Jb_S_L3 + Jb_S_L4.'*M4*Jb_S_L4);

% Lagrangian Coriolis matrix
CR = sym(zeros(size(MR)));
for i = 1 : size(MR, 1)
    for j = 1 : size(MR, 2)
        CR(i, j) = simplify(christoffel2(MR, q, dq, i, j));
    end
end

% Potential energy
g = 9.81;    % Gravitational acceleration constant - m/s^2
VR = m2L*g*theta2*cos(q2) + ...
    m3L*g*(d2*cos(q2)+theta3*cos(q2+q3)) + ...
    m4L*g*(d2*cos(q2)+d3*cos(q2+q3)+theta4*cos(q2+q3+q4));

% Lagrangian nonlinear matrix
NR = simplify([diff(VR, q2) diff(VR, q3) diff(VR, q4)]).';

% Joint torques
syms tau2 tau3 tau4 real;
tau = [tau2 tau3 tau4].';
UR = tau;

% % Convert dynamic matrices into functions
% % NOTE: If regenerating, need to edit each before use.
% cmt = 'See Dynamics_2D.mat for derivation.';
% MR_2Dfunc = matlabFunction(MR, 'Comments', cmt, 'File', 'MR_2Dfunc.m', 'Outputs', {'M'});
% CR_2Dfunc = matlabFunction(CR, 'Comments', cmt, 'File', 'CR_2Dfunc.m', 'Outputs', {'C'});
% NR_2Dfunc = matlabFunction(NR, 'Comments', cmt, 'File', 'NR_2Dfunc.m', 'Outputs', {'N'});
% UR_2Dfunc = matlabFunction(UR, 'Comments', cmt, 'File', 'UR_2Dfunc.m', 'Outputs', {'U'});


%% Test forward kinematics
clc
test1 = double(subs(g_P_T, q, [0 0 0]'));
true1 = [0 -1 d0+d1; 1 0 h1+d2+d3+d4; 0 0 1];
pass1 = all(abs(test1 - true1) < 1e-10, 'all')

test2 = double(subs(g_P_T, q, [0 pi/2 0]'));
true2 = [-1 0 d0+d1-d3-d4; 0 -1 h1+d2; 0 0 1];
pass2 = all(abs(test2 - true2) < 1e-10, 'all')

test3 = double(subs(g_P_T, q, [-pi/2 pi/2 -pi/2]'));
true3 = [1 0 d0+d1+d2+d4; 0 1 h1+d3; 0 0 1];
pass3 = all(abs(test3 - true3) < 1e-10, 'all')

test4 = double(subs(g_P_T, q, [pi -pi pi]'));
true4 = [0 1 d0+d1; -1 0 h1-d2+d3-d4; 0 0 1];
pass4 = all(abs(test4 - true4) < 1e-10, 'all')
```

```
%% Simulate dynamics
% Arm initial conditions
q0 = [-pi/6 -pi/6 -pi/3].';
w0 = [0 0 0].';
x0 = [q0; w0];

% Arm simulation time
t0 = 0;
tf = t0 + 8;

% Run arm simulation
[tRsim, xRsim] = ode45(@arm_odedyn, (t0 : 0.01 : tf), x0);

% Animate platform and arm
animate_2D(0, 0, tRsim, xRsim(:, 1:3));


%% Helper functions for dynamics solution
function [Gamma] = christoffel2(M, q, dq, i, j)
    Gamma = 0;
    for k = 1 : length(q)
        dMijdqk = diff(M(i, j), q(k));
        dMikdqj = diff(M(i, k), q(j));
        dMkjdqi = diff(M(k, j), q(i));
        Gamma = Gamma + (dMijdqk + dMikdqj - dMkjdqi) * dq(k) / 2;
    end
end

function [x_vee] = rbvel2twist2D(x)
    v = x(1:2, 3);
    w_hat = x(1:2, 1:2);
    w = w_hat(2, 1);

    x_vee = [v; w];
end


%% Helper functions for simulation
function[dx] = arm_odedyn(t, x)
    q = x(1:3);
    dq = x(4:6);

    M = MR_2Dfunc(q);
    C = CR_2Dfunc(q, dq);
    N = NR_2Dfunc(q);
    tau = arm_odedyninputs(t);
    U = UR_2Dfunc(tau);

    b = 0.1 * eye(3);    % Arbitrary joint damping
    ddq = M \ (U - C*dq - N - b*dq);
    dx = [dq; ddq];
end

function[tau] = arm_odedyninputs(t)
    tau = [0 0 0].';
end
```

```
%{
animate_2D.m
2D dynamics animation for 3-link robotic arm and mobile base
%}

function[] = animate_2D(tM, y, tR, q)

% Link lengths (m)
global d0 d1 h1 d2 d3 d4;
d0 = 151 / 1000;
d1 = 33 / 1000;
h1 = 147 / 1000;
d2 = 155 / 1000;
d3 = 135 / 1000;
d4 = 217.5 / 1000;

dt = 0.02;

f = figure();
set(f, 'Position', [1 41 1536 748.8]);%get(0, 'Screensize'));
set(f, 'DoubleBuffer', 'on');
hold on;
axis equal;
axis([-1 7 -1 1]);
[r1, w1] = drawplatform(y(1));
[p1, p2, p3, p4, p5] = drawarm(y(1), q(1, :));
[t1] = drawtime(tM(1));

disp('Press any key to animate...');
pause;

% Animate platform
for i = 1 : length(tM)
    start = tic;

    % Clear previous plots.
    delete(r1);
    delete(w1);
    delete(p1);
    delete(p2);
    delete(p3);
    delete(p4);
    delete(p5);
    delete(t1);

    % Plot new
    [r1, w1] = drawplatform(y(i));
    [p1, p2, p3, p4, p5] = drawarm(y(i), q(1, :));
    [t1] = drawtime(tM(i));
    drawnow limitrate;

    finish = toc(start);
    if(finish < dt)
        pause(dt - finish);
    end
end

% Animate arm
for i = 1 : length(tR)
    start = tic;

    % Clear previous plots.
    delete(p1);
    delete(p2);
    delete(p3);
```

```
    delete(p4);
    delete(p5);
    delete(t1);

    % Plot new
    [p1, p2, p3, p4, p5] = drawarm(y(end), q(i, :));
    [t1] = drawtime(tR(i));
    drawnow limitrate;

    finish = toc(start);
    if(finish < dt)
        pause(dt - finish);
    end
end


disp('Press any key to close figure...');
pause;
close(f);
end

% Plot arm
function[p1, p2, p3, p4, p5] = drawarm(y0, q)
    global d0 d1 h1 d2 d3 d4;

    z0 = 0;
    q2 = q(1);
    q3 = q(2);
    q4 = q(3);

    y2P = y0 + d0 + d1;
    z2P = z0 + h1;

    y2S = z2P;
    z2S = -y2P;
    y3 = y2S + d2 * cos(q2);
    z3 = z2S + d2 * sin(q2);
    y4 = y3 + d3 * cos(q2+q3);
    z4 = z3 + d3 * sin(q2+q3);
    yt = y4 + d4 * cos(q2+q3+q4);
    zt = z4 + d4 * sin(q2+q3+q4);

    p1 = plot([y0 y2P], [z0 z2P], '--k');
    p2 = plot(-[z2S z3 z4 zt], [y2S y3 y4 yt], '-k', 'LineWidth', 1);   % links
    p3 = plot(y0, z0, '^k', 'MarkerSize', 4, 'MarkerFaceColor', 'k');   % base joint
    p4 = plot(-[z2S z3 z4], [y2S y3 y4], '.b', 'MarkerSize', 8);        % intermediate joints
    p5 = plot(-zt, yt, '.r', 'MarkerSize', 12);                          % end-effector
end

% Plot platform
function[r1, w1] = drawplatform(y)
    r1 = rectangle('Position', [y-0.29 -0.14 0.58 0.14], 'FaceColor', [1 0.6 0]);
    w1 = plot(y+[-0.2355 0.2355], [-0.12 -0.12], '.', 'MarkerSize', 35, 'Color', [0.5 0.5 0.5]);
end

% Display time
function[t1] = drawtime(t)
    t1 = text(-1, 0.85, strcat(num2str(t), ' s'), 'FontSize', 12);
end
```